

DIFFERENTIAL EIGENFORMS

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ABSTRACT. The aim of this paper is to show how δ -characters of Abelian varieties (in the sense of [3]) can be used to construct δ -modular forms of weight 0 and order 2 (in the sense of [5]) which are eigenvectors of Hecke operators. These δ -modular forms have “essentially the same” eigenvalues as certain classical complex eigenforms of weight 2.

1. INTRODUCTION

The concept of δ -modular form was introduced in [5]. Very roughly speaking a level one δ -modular form of order r is a “homogeneous” function of plane elliptic curves $y^2 = x^3 + ax + b$ (where $a, b \in R := \hat{\mathbf{Z}}_p^{ur}$) that can be written as a p -adic restricted power series in $a, b, \delta a, \delta b, \dots, \delta^r a, \delta^r b, \Delta^{-1}$, where $\Delta := 4a^3 + 27b^2$ and $\delta^i a, \delta^i b$ are the iterated “Fermat quotients” of a, b with respect to p . We recall that $\delta x := (\phi(x) - x^p)/p$, where $\phi : R \rightarrow R$ is the lift of the p -power Frobenius on R/pR . Morally one may view δ as an arithmetic analogue of a derivation (acting on “numbers” rather than “functions”) and one may view δ -modular forms as “non-linear arithmetic differential operators of order r ” acting on pairs (a, b) . We shall review this concept presently, from a slightly different (but equivalent) viewpoint. There is a level N generalization of this. Also there are Hecke operators $T(l)$ acting on δ -modular forms (where l are primes with $(l, Np) = 1$) so one can talk about δ -eigenforms (for all these $T(l)$ ’s). Finally one can attach, to δ -modular forms of order r , δ -Fourier expansions which are series in the variables $q, q', \dots, q^{(r)}$. For applications of our theory we refer to [5], [6].

There is an “easy” way to construct δ -eigenforms by considering I -linear combinations of “ ϕ -powers”, f^{ϕ^j} , of classical (complex) eigenforms f where I is the ring generated by the isogeny covariant δ -modular forms (in a sense generalizing that in [5]). A natural question is whether all δ -eigenforms can be obtained in this way. As we shall see in this paper the answer is *no*. Indeed, we provide, in this paper, a construction of δ -eigenforms $f^\#$ of weight 0 and order 2 that have “essentially the same” Hecke eigenvalues as certain classical eigenforms f of weight 2 (and order 0). As we shall see, forms of weight 0 (such as $f^\#$) are never I -linear combinations of forms f^{ϕ^j} . Having constructed the forms $f^\#$ one can ask, of course, if any δ -eigenform is an I -linear combination of forms f^{ϕ^j} and $(f^\#)^{\phi^i}$; at this point it is not clear what to expect.

The δ -Fourier expansion of $f^\#$ will be related in an interesting way to the Fourier expansion of f . Indeed, if $f = \sum a_n q^n$ is a (classical) newform of weight 2 on $\Gamma_0(N)$ (which is not of “CM type”) with Fourier coefficients $a_n \in \mathbf{Z}$, then the δ -Fourier

expansion of f^\sharp will be a series $f_\infty^\sharp(q, q', q'')$ in 3 variables q, q', q'' which, after the substitution $q' = q'' = 0$, becomes equal to the series

$$f^{(-1)}(q) := \sum_{(n,p)=1} \frac{a_n}{n} q^n.$$

(A similar, but more complicated statement holds for f of “CM type”.) The series $f^{(-1)}$ is, of course, not the Fourier series of any (classical) eigenform but, rather, a p -adic modular form in the sense of Serre; cf. [21], p. 115. Note that, viewed as a function of elliptic curves in the sense of Katz [16] the p -adic modular form $f^{(-1)}$ does not extend across the “supersingular disks” because, if this were the case, $f^{(-1)}$ would define a non-constant function on a projective modular curve. On the other hand, remarkably, the δ -modular form f^\sharp does extend across the “supersingular disks” (this being the case with *any* δ -modular form). One may ask if, in spite of this phenomenon, f^\sharp is, nevertheless, a linear combination, with isogeny covariant coefficients defined outside the supersingular disks, of ϕ -powers of $f^{(-1)}$; we will show that this is not the case.

The idea in our construction of the forms f^\sharp is to use the Eichler-Shimura construction for the f 's in conjunction with our theory of δ -characters introduced in [3]. (Roughly speaking δ -characters are homomorphisms from the group of R -points of an Abelian variety to the additive group of R which, in coordinates, are given by expressions involving the coordinates of the points and their iterated Fermat quotients. They are arithmetic analogues of the *Manin maps* introduced by Manin in the context of the Mordell conjecture over function fields [14].) Then our forms f^\sharp will arise by composing certain δ -characters of the modular Jacobians $J_1(N)$ with the Abel-Jacobi maps $X_1(N) \rightarrow J_1(N)$ that send a fixed cusp into 0.

Here is the plan of this paper. In Section 2 we review (and slightly extend) the concept of δ -modular form and Hecke operators in [5]. Then we state one of our main results about the existence of the forms f^\sharp and their independence from f . In Section 3 we review results of Eichler-Shimura and Manin-Drinfeld. Section 4 reviews δ -characters [3] and examines the existence of eigenvectors in the space of δ -characters. In Section 5 we conclude our construction of the forms f^\sharp ; it will turn out that the forms f^\sharp “vanish at all the cusps”. In Section 6 we introduce δ -Fourier expansions at ∞ and we compute them for our forms f^\sharp . In Section 7 we use δ -Fourier expansions to prove, in particular, the independence of f^\sharp from $f^{(-1)}$. In Section 8 we use δ -Fourier expansions to compute the effect of δ -Serre operators (in the sense of [6]) on f^\sharp . In Section 9 we prove that the δ -Fourier expansion of f^\sharp is in the domain of definition of the (partially defined) Hecke operator $T(p)_\infty$ and is an eigenvector of this operator. We end the paper by stating a result (whose proof will be given in a subsequent paper [7]) saying that δ -modular forms which “vanish at the cusps” and are in the domain of definition of $T(p)_\infty$ automatically arise from composing δ -characters of the modular Jacobians with Abel-Jacobi maps. This is, in some sense, a converse of our existence results for the forms f^\sharp in the present paper.

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2. MAIN CONCEPTS

2.1. Prolongation sequences. Our main reference here is [5]. We fix, throughout this paper, a prime integer $p \geq 5$. Let $C_p(X, Y) \in \mathbf{Z}[X, Y]$ be the polynomial with integer coefficients

$$C_p(X, Y) := \frac{X^p + Y^p - (X + Y)^p}{p}.$$

A p -derivation from a ring A into an A -algebra $\varphi : A \rightarrow B$ is a map $\delta : A \rightarrow B$ such that $\delta(1) = 0$ and

$$\begin{aligned} \delta(x + y) &= \delta x + \delta y + C_p(\varphi(x), \varphi(y)) \\ \delta(xy) &= \varphi(x)^p \cdot \delta y + \varphi(y)^p \cdot \delta x + p \cdot \delta x \cdot \delta y, \end{aligned}$$

for all $x, y \in A$. Given a p -derivation we always denote by $\phi : A \rightarrow B$ the map $\phi(x) = \varphi(x)^p + p\delta x$; then ϕ is a ring homomorphism. A *prolongation sequence* is a sequence S^* of rings S^n , $n \geq 0$, together with ring homomorphisms $\varphi_n : S^n \rightarrow S^{n+1}$ and p -derivations $\delta_n : S^n \rightarrow S^{n+1}$ such that $\delta_{n+1} \circ \varphi_n = \varphi_{n+1} \circ \delta_n$ for all n . We usually denote all φ_n by φ and all δ_n by δ and we view S^{n+1} as an S^n -algebra via φ . A morphism of prolongation sequences, $u^* : S^* \rightarrow \tilde{S}^*$ is a sequence $u^n : S^n \rightarrow \tilde{S}^n$ of ring homomorphisms such that $\delta \circ u^n = u^{n+1} \circ \delta$ and $\varphi \circ u^n = u^{n+1} \circ \varphi$. Let W be the ring of polynomials $\mathbf{Z}[\phi]$ in the indeterminate ϕ . Then, for $w = \sum_{i=0}^r a_i \phi^i \in W$, we set $\deg(w) := \sum a_i$. If $a_r \neq 0$ we set $\text{ord}(w) = r$; we also set $\text{ord}(0) = 0$. For w as above (respectively for $w \in W_+ := \{\sum b_i \phi^i \mid b_i \geq 0\}$), S^* a prolongation sequence, and $x \in (S^0)^\times$ (respectively $x \in S^0$) we can consider the element $x^w := \prod_{i=0}^r \varphi^{r-i} \phi^i(a)^{a_i} \in (S^r)^\times$ (respectively $x^w \in S^r$). We let $W(r) := \{w \in W \mid \text{ord}(w) \leq r\}$.

Let $R := R_p := \hat{\mathbf{Z}}_p^{ur}$ be the completion of the maximum unramified extension of \mathbf{Z}_p . Then R has a unique p -derivation $\delta : R \rightarrow R$ given by $\delta x = (\phi(x) - x^p)/p$ where $\phi : R \rightarrow R$ is the unique lift of the p -power Frobenius map on R/pR . One can consider the prolongation sequence R^* where $R^n = R$ for all n . By a *prolongation sequence over R* we understand a prolongation sequence S^* equipped with a morphism $R^* \rightarrow S^*$. From now on all our prolongation sequences are assumed to be over R .

2.2. δ -modular forms. Our main reference here is, again, [5]. We fix, throughout this paper, an integer $N \geq 1$, not divisible by p . For any ring S let us denote by $\mathbf{M}(\Gamma_1(N), S)$ the set of all triples $(E/S, \alpha, \omega)$ where E/S is an elliptic curve, ω is an invertible 1-form on E , and $\alpha : (\mathbf{Z}/N\mathbf{Z})_S \rightarrow E$ is a closed immersion of group schemes (referred to as a $\Gamma_1(N)$ -level structure). Fix $w \in W$ with $\text{ord}(w) \leq r$. A δ -modular form of weight $w \in W$ and order r on $\Gamma_1(N)$ is a rule f that associates to any prolongation sequence S^* of Noetherian, p -adically complete rings and any triple $(E/S^0, \alpha, \omega) \in \mathbf{M}(\Gamma_1(N), S^0)$ an element $f(E/S^0, \alpha, \omega, S^*) \in S^r$ such that the following properties are satisfied:

- (1) $f(E/S^0, \alpha, \omega, S^*)$ depends on the isomorphism class of $(E/S^0, \alpha, \omega)$ only.
- (2) Formation of $f(E/S^0, \alpha, \omega, S^*)$ commutes with base change $u^* : S^* \rightarrow \tilde{S}^*$ i.e.

$$f(E \otimes_{S^0} \tilde{S}^0 / \tilde{S}^0, \alpha \otimes \tilde{S}^0, u^{0*} \omega, \tilde{S}^*) = u^r(f(E/S^0, \alpha, \omega, S^*)).$$

- (3) $f(E/S^0, \alpha, \lambda\omega, S^*) = \lambda^{-w} \cdot f(E/S^0, \alpha, \omega, S^*)$ for all $\lambda \in (S^0)^\times$.

We denote by $M^r(\Gamma_1(N), R, w)$ the R -module of all δ -modular forms over R of weight $w \in W$ and order r on $\Gamma_1(N)$. Then the direct sum

$$M^r(\Gamma_1(N), R, *) := \bigoplus_{w \in W(r)} M^r(\Gamma_1(N), R, w)$$

has a natural structure of graded ring. We view $M^r(\Gamma_1(N), R, *)$ as a subring of $M^{r+1}(\Gamma_1(N), R, *)$ via φ and we have naturally induced ring homomorphisms $\phi : M^r(\Gamma_1(N), R, *) \rightarrow M^{r+1}(\Gamma_1(N), R, *)$ sending any $f \in M^r(\Gamma_1(N), R, w)$ into $f^\phi := \phi \circ f \in M^{r+1}(\Gamma_1(N), R, \phi w)$; for $w = \sum a_i \phi^i \in W_+$ we write $f^w := \prod (f^{\phi^i})^{a_i}$. The rings $M^r(\Gamma_1(N), R, *)$ are integral domains. Their union will be denoted by $M^\infty(\Gamma_1(N), R, *)$.

We end our discussion here by noting that, by [6], p.252, the spaces $M^r(\Gamma_1(N), R, w)$ embed into spaces of *ordinary δ -modular forms*, denoted by

$$M_{ord}^r(\Gamma_1(N), R, w)$$

and defined exactly as the spaces $M^r(\Gamma_1(N), R, w)$ except that instead of the set $\mathbf{M}(\Gamma_1(N), S^0)$ one considers the set $\mathbf{M}_{ord}(\Gamma_1(N), S^0)$ of all tuples in $\mathbf{M}(\Gamma_1(N), S^0)$ with ordinary reduction.

2.3. δ -Hecke operators. Again, our main reference here is [5]. Assume S^* is a prolongation sequence of Noetherian, p -adically complete rings, and let \tilde{S} be a finite étale over-ring of S^0 . Then, by [5], (1.6), there is a unique structure of prolongation sequence on $S^* \otimes_{S^0} \tilde{S} := (S^n \otimes_{S^0} \tilde{S})$ compatible (in the obvious sense) with that of S^* . Now let l be a prime integer not dividing Np . Let $f \in M^r(\Gamma_1(N), R, w)$ be a δ -modular form. We can define a δ -modular form $T(l)f \in M^r(\Gamma_1(N), R, w)$ by the formula

$$(2.1) \quad (T(l)f)(E/S^0, \alpha, \omega, S^*) = \sum_{i=0}^l f(\tilde{E}_i/\tilde{S}, u_i \circ \alpha, u_{i*}\omega, S^* \otimes_{S^0} \tilde{S})$$

where \tilde{S} is any finite étale over-ring of S^0 such that the group scheme of points of order l of $\tilde{E} := E \otimes_{S^0} \tilde{S}$ is isomorphic to $(\mathbf{Z}/l\mathbf{Z})_{\tilde{S}}^2$ (hence the elliptic curve \tilde{E} has exactly $l+1$ finite, flat subgroup schemes H_0, \dots, H_l of rank l), $\tilde{E}_i = \tilde{E}/H_i$, $u_i : \tilde{E} \rightarrow \tilde{E}_i$ are the natural projections, and the $u_{i*}\omega$'s are induced by ω via pull-back to \tilde{E} followed by trace to the \tilde{E}_i 's. In the above we can always assume \tilde{S} is Galois over S^0 . Note that $(T(l)f)(E/S^0, \alpha, \omega, S^*)$ which is, a priori, an element of $S^r \otimes_{S^0} \tilde{S}$, actually belongs to S^r , and does not depend on the choice of \tilde{S} .

We refer to the maps $T(l) : M^r(\Gamma_1(N), R, w) \rightarrow M^r(\Gamma_1(N), R, w)$ as *δ -Hecke operators*. Clearly these maps commute with ϕ . For $r=0$ and $w=m \in \mathbf{Z}$ one can normalize our $T(l)$ in the classical fashion by considering the operators

$$T_m(l) := l^{m-1}T(l) : M^0(\Gamma_1(N), R, m) \rightarrow M^0(\Gamma_1(N), R, m).$$

2.4. Classical eigenforms. Our main references here are [20, 8]. Denote by $S_m(\Gamma_1(N), \mathbf{C})$ the space of (classical) cusp forms of weight m on $\Gamma_1(N)$ over the complex field \mathbf{C} . On this space one has Hecke operators $T_m(n)$ acting, $n \geq 1$. An *eigenform* $f \in S_m(\Gamma_1(N), \mathbf{C})$ is a nonzero element which is a simultaneous eigenvector for all $T_m(n)$, $n \geq 1$. An eigenform $f = \sum_{n \geq 1} a_n q^n$, $a_n = a_n(f)$, is *normalized* if $a_1 = 1$; in this case $T_m(n)f = a_n \cdot f$ for all $n \geq 1$. One associates to any eigenform $f \in S_m(\Gamma_1(N), \mathbf{C})$ its *system of eigenvalues* $l \mapsto a_l$, $(l, N) = 1$. A *newform* is a

normalized eigenform whose system of eigenvalues does not come from a system of eigenvalues associated to an eigenform in $S_m(\Gamma_1(M), \mathbf{C})$ with $M \mid N$, $M \neq N$. For any normalized eigenform $f \in S_m(\Gamma_1(N), \mathbf{C})$ one may consider the subring \mathcal{O}_f of \mathbf{C} generated by all $a_n(f)$, $n \geq 1$; then \mathcal{O}_f is a finite \mathbf{Z} -algebra and one denotes by K_f its fraction field. If $Q \geq 1$ is any integer we denote by $\mathcal{O}_f^{(Q)}$ the subring of \mathbf{C} generated by all $a_l(f)$, where l is prime, not dividing Q .

We will later need to consider the subspace $S_m(\Gamma_0(N), \mathbf{C})$ of $S_m(\Gamma_1(N), \mathbf{C})$ of all cusp forms of weight m on $\Gamma_0(N)$. Recall that if $f = \sum a_n q^n$ is an eigenform in $S_m(\Gamma_0(N), \mathbf{C})$ then

$$(2.2) \quad \begin{aligned} a_{n_1 n_2} &= a_{n_1} a_{n_2} \text{ for } (n_1, n_2) = 1 \\ a_{l^{i-1} a_l} &= a_{l^i} + l^{m-1} a_{l^{i-2}} \text{ for } l \text{ prime, } (l, N) = 1 \text{ and } i \geq 2, \\ a_{l^{i-1} a_l} &= a_{l^i} \text{ for } l \text{ prime, } l \mid N \text{ and } i \geq 2. \end{aligned}$$

2.5. δ -eigenforms. A non-zero δ -modular form $h \in M^r(\Gamma_1(N), R_p, w)$ is called a δ -eigenform if $T(l)h = \lambda_l \cdot h$, $\lambda_l \in R_p$, for all primes l not dividing Np . A δ -eigenform is said to *belong* (outside Np) to a (classical) normalized eigenform $f = \sum a_n q^n \in S_m(\Gamma_1(N), \mathbf{C})$ if there exist a (necessarily injective) ring homomorphism $\chi : \mathcal{O}_f^{(Np)} \rightarrow R_p$ and an integer $e \in \mathbf{Z}$ such that $\lambda_l = l^e \chi(a_l)$ for all primes l not dividing Np . We then say that $f^\#$ belongs to f with *character* χ and *exponent* e .

Note that χ is uniquely determined by $f^\#$. Indeed assume $\chi, \chi' : \mathcal{O}_f^{(Np)} \rightarrow R_p$ are ring homomorphisms and e, e' are integers such that $l^e \chi(a_l) = l^{e'} \chi'(a_l)$ for all primes l not dividing Np and assume $\chi \neq \chi'$. Then clearly $e \neq e'$. Set $L := l^{e-e'} \neq 1$ and, since $\chi \neq \chi'$, one can choose a prime l not dividing Np such that $a_l \neq 0$. Let $\Phi(t) = t^d + b_1^{d-1} + \dots + b_d \in \mathbf{Q}[t]$ be the minimal polynomial of a_l over \mathbf{Q} . Then both $\chi(a_l)$ and $L \cdot \chi(a_l)$ are roots of $\Phi(t)$. Hence $\chi(a_l)$ is a root of $\Psi(t) := \Phi(t) - L^{-d} \Phi(Lt) = \sum_{i=1}^d b_i (1 - L^{-i}) t^{d-i}$. Since $\Psi(t)$ has degree $\leq d-1$ we must have $\Psi(t) = 0$ hence $\Phi(t) = t$ hence $a_l = 0$, a contradiction.

2.6. δ -eigenforms arising from classical eigenforms. There is an “easy” way to construct δ -eigenforms belonging to classical eigenforms f by taking linear combinations of “ ϕ -powers of f ” with *isogeny covariant* δ -modular forms (in a sense slightly generalizing that in [5]). In what follows we explain this construction. We should point out that the forms $f^\#$ mentioned in the Introduction will be shown not to be obtainable via this construction.

Let $F \in M^r(\Gamma_1(N), R, w)$ be a δ -modular form of weight $w = \sum n_i \phi^i$ on $\Gamma_1(N)$. Assume $\deg(w) := \sum n_i$ is even. Generalizing the level one definition in [5] we say that F is *isogeny covariant* if for any prolongation sequence S^* , any triples $(E_1, \alpha_1, \omega_1), (E_2, \alpha_2, \omega_2) \in \mathbf{M}(\Gamma_1(N), S^0)$, and any isogeny $u : E_1 \rightarrow E_2$ of degree prime to p , with $\omega_1 = u^* \omega_2$ and $u \circ \alpha_1 = \alpha_2$, we have

$$F(E_1, \alpha_1, \omega_1, S^*) = \deg(u)^{-\deg(w)/2} \cdot F(E_2, \alpha_2, \omega_2, S^*).$$

Example 2.1. By [6], p. 268 and Theorem 8.83, for each $r \geq 1$ the R_p -module of isogeny covariant δ -modular forms in $M^r(\Gamma_1(N), R_p, -1 - \phi^r)$ is free of rank one. Following [6] we shall denote by $f^r = f_{crys}^r$ a basis of this rank one module. (So the upper r is an index, not an exponent. Recall from [6] that f^r is constructed via crystalline cohomology.)

We denote by $\mathcal{I} \subset M^\infty(\Gamma_1(N), R, *)$ the multiplicative system of all non-zero isogeny covariant δ -modular forms and by $\mathcal{J} \subset \mathcal{I}$ the multiplicative system generated by all $(f^r)^{\phi^s}$ for $r \geq 1$ and $s \geq 0$. The R -linear spans of \mathcal{I} and \mathcal{J} will be denoted by I and J respectively. Then I is a ring, J is a subring of I , and it is tempting to conjecture [1, 2] that $J \otimes \mathbf{Q} = I \otimes \mathbf{Q}$.

Lemma 2.2. *If $F \in M^r(\Gamma_1(N), R, w)$ is isogeny covariant and*

$$G \in M^r(\Gamma_1(N), R, v)$$

is any δ -modular form then, for any prime l not dividing Np ,

$$T(l)(F \cdot G) = l^{-\deg(w)/2} \cdot F \cdot T(l)G.$$

In particular, if G is a δ -eigenform belonging to the classical normalized eigenform $f \in S_m(\Gamma_1(N), \mathbf{C})$ with character χ and exponent e then $F \cdot G$ is a δ -eigenform belonging to f with character χ and exponent $e - \frac{\deg(w)}{2}$.

Proof. This follows from a computation similar to the one in [3], p.125 (where the case $N = 1$ was treated). \square

Now let $f \in S_m(\Gamma_1(N), \mathbf{C})$, $f = \sum a_n q^n$, be a normalized eigenform of weight m and let $\rho : \mathcal{O}_f[1/N, \zeta_N] \rightarrow R_p$ be any ring homomorphism, where p does not divide N . Then, by the “ q -expansion principle” [10], pp. 70 and 112, f naturally defines (via ρ) a rule f^ρ (compatible with base change and homogeneous of degree $-m$) that attaches to any R_p -algebra S and any triple $(E/S, \alpha, \omega) \in \mathbf{M}(\Gamma_1(N), S)$ an element $f^\rho(E/S, \alpha, \omega) \in S$ depending only on the isomorphism class of the triple. (Here it is essential that we have a fixed primitive N -th root of unity $\rho(\zeta_N)$ in R_p .) Then f^ρ induces a δ -modular form (still denoted by) $f^\rho \in M^0(\Gamma_1(N), R_p, m)$, of weight m , defined by the formula

$$f^\rho(E/S^0, \alpha, \omega, S^*) := f^\rho(E/S^0, \alpha, \omega).$$

The composition $f^{\rho\phi^j} := \phi^j \circ f^\rho$ is a well defined element of $M^j(\Gamma_1(N), R_p, \phi^j w)$. There is an obvious compatibility between the classical and our Hecke operators $T_m(l)$, which yields:

$$\begin{aligned} (2.3) \quad T(l)f^{\rho\phi^j} &= (T(l)f^\rho)^{\phi^j} \\ &= (l^{1-m}T_m(l)f^\rho)^{\phi^j} \\ &= l^{1-m}(T_m(l)f)^{\rho\phi^j} \\ &= l^{1-m}\phi^j(\rho(a_l)) \cdot f^{\rho\phi^j} \end{aligned}$$

for all primes l not dividing Np . So we see that $f^{\rho\phi^j}$ is a δ -eigenform of order j and weight $m\phi^j$ which belongs to f with character $\phi^j \circ \rho$ and exponent $e = 1 - m$.

In particular, according to Lemma 2.2, if $\phi^b \circ \rho = \rho$ for some $b \geq 1$, and if $a \geq 0$, then any δ -modular function of weight w in the I -linear span of

$$\{f^{\rho\phi^a}, f^{\rho\phi^{a+b}}, f^{\rho\phi^{a+2b}}, f^{\rho\phi^{a+3b}}, \dots\}$$

is a δ -eigenform belonging to f with character $\phi^a \circ \rho$ and exponent

$$(2.4) \quad e = 1 - \frac{m + \deg(w)}{2}.$$

Lemma 2.3. *Let $f \in S_2(\Gamma_1(N), \mathbf{C})$ be a normalized eigenform and let \tilde{f} be a non-zero δ -modular form of weight 0. Then \tilde{f} cannot be in the I -linear span of*

$$\{f^\rho, f^{\rho\phi}, f^{\rho\phi^2}, f^{\rho\phi^3}, \dots\}.$$

Proof. Assume the conclusion is false. We may assume $\tilde{f} = \sum F_a \cdot f^{\rho\phi^a}$, where F_a are isogeny covariant of weight $-2\phi^a$. To get a contradiction we need to check the following:

Claim. *For any $0 \leq a \leq r$ there are no non-zero isogeny covariant δ -modular forms in $M^r(\Gamma_1(N), R_p, -2\phi^a)$.*

We fix a and prove this claim by induction on r . Assume first $r = a$. If $a = 0$ then the claim follows from [6], Proposition 8.75. If $a \geq 1$ our claim follows from [6], Theorem 8.83, assertion 2. To perform the induction step assume $r > a$. Then, by [6], Corollary 8.40 and Proposition 8.75, if $h \in M^r(\Gamma_1(N), R_p, -2\phi^a)$ is isogeny covariant then $\partial_r h = 0$ where ∂_r is the δ -Serre operator in loc. cit. This easily implies that $h \in M^{r-1}(\Gamma_1(N), R_p, -2\phi^a)$ and we conclude by the induction hypothesis. (The various results in [6] quoted above apply to our situation in view of [6], Proposition 8.22. Note also that the special case $N = 1$ of our claim was proved by Barcau [1].) \square

2.7. Ordinary δ -modular forms arising from p -adic modular forms. The main references here are [21, 16, 12]. Let $g \in \mathbf{Q}_p[[q]]$ be a p -adic modular form of weight $m \in \mathbf{Z}$ in the sense of Serre. Fix a homomorphism $\rho : \mathbf{Z}[\zeta_N, 1/N] \rightarrow R_p$. Then g induces a p -adic modular form g^ρ of level N , weight m and growth 1 in the sense of Katz [16]; cf. [12], Theorem 6.21, p. 158. On the other hand g^ρ induces an ordinary δ -modular form (still denoted by) $g^\rho \in M_{ord}^0(\Gamma_1(N), R_p, m)$. So for each $j \geq 0$ we may consider the ordinary δ -modular form $g^{\rho\phi^j} \in M_{ord}^j(\Gamma_1(N), R_p, m\phi^j)$.

In particular, if $f = \sum a_n q^n \in S_2(\Gamma_1(N), \mathbf{C})$ is a normalized eigenform of weight 2 with $a_n \in \mathbf{Z}$ then, by [21], p. 115, the series

$$f^{(-1)} = \sum_{(n,p)=1} \frac{a_n}{n} q^n$$

is a p -adic modular form of weight 0 such that $T_0(l)f^{(-1)} = l^{-1}a_l f^{(-1)}$ for l prime different from p . It immediately follows that

$$f^{(-1)\rho\phi^j} \in M_{ord}^j(\Gamma_1(N), R_p, 0)$$

is, in the obvious sense, an ordinary δ -eigenform belonging to f with exponent 0. So any R -linear combination of such forms will have the same property.

Note that, if in the definition of isogeny covariant δ -modular forms, one replaces $\mathbf{M}(\Gamma_1(N), S^0)$ by $\mathbf{M}_{ord}(\Gamma_1(N), S^0)$ one obtains the notion of *ordinary isogeny covariant δ -modular form*. Let \mathcal{I}_{ord} be the multiplicative system of all such forms and let I_{ord} be the R -linear span of \mathcal{I}_{ord} . Then I_{ord} is a ring.

Lemma 2.4. *Let $f \in S_2(\Gamma_1(N), \mathbf{C})$ be a normalized eigenform of weight 2 with $a_n \in \mathbf{Z}$ and let \tilde{f} be an ordinary δ -modular form of weight 0 which is in the I_{ord} -linear span of the set*

$$\{f^{(-1)\rho}, f^{(-1)\rho\phi}, f^{(-1)\rho\phi^2}, f^{(-1)\rho\phi^3}, \dots\}.$$

Then \tilde{f} is in the R -linear span of this set.

Proof. Let $\tilde{f} = \sum_j F_j \cdot f^{(-1)\rho\phi^j}$, $F_j \in I_{ord}$. Picking the weight 0 components we may assume F_j have weight 0. So we are reduced to showing that any ordinary isogeny covariant δ -modular form of weight 0 is a constant in R . This follows from [4], Propositions 7.21 (plus the Remark after it) and 7.23. \square

2.8. The forms f^\sharp . The main purpose of this paper is to provide a construction of δ -eigenforms f^\sharp of weight 0 and order 2 belonging to classical eigenforms f of weight 2. As we shall see, the forms f^\sharp will be neither I -linear combinations of ϕ -powers of f nor I_{ord} -linear combinations of ϕ -powers of $f^{(-1)}$. Here is one of our main results. This result will be complemented by other results later in the paper; cf. Remark 2.6 below.

Theorem 2.5. *Let $f \in S_2(\Gamma_1(N), \mathbf{C})$ be a newform of weight 2 on $\Gamma_1(N)$, $N > 4$, and let $g := [K_f : \mathbf{Q}]$. Then, for any sufficiently large prime p , and any embedding $\rho : \mathcal{O}_f[1/N, \zeta_N] \rightarrow R_p$, there exist δ -eigenforms*

$$f_1^\sharp, \dots, f_g^\sharp \in M^2(\Gamma_1(N), R_p, 0),$$

of weight 0 and order 2 on $\Gamma_1(N)$, such that

- 1) f_j^\sharp belongs to f with exponent 0,
- 2) f_j^\sharp is not in the I -linear span of $\{f^\rho, f^{\rho\phi}, f^{\rho\phi^2}, f^{\rho\phi^3}, \dots\}$,
- 3) $f_1^\sharp, \dots, f_g^\sharp$ are R_p -linearly independent.

Remark 2.6. 1) Assertion 2 follows directly from Lemma 2.3.

2) As we shall see, f_j^\sharp themselves should be morally viewed as a kind of “ δ -cusp forms” in the sense that they “vanish at the cusps”; cf. Remark 5.1.

3) In case $g = 1$ (i.e. $a_n(f) \in \mathbf{Q}$, equivalently $a_n(f) \in \mathbf{Z}$) the form $f^\sharp = f_1^\sharp$ will be essentially canonically associated to f and its δ -Fourier expansion will be closely related to that of f ; cf. Theorems 6.3 and 6.5. We will show that f^\sharp is not an R_p -linear combination of ϕ -powers of $f^{(-1)}$; cf. Theorem 7.1. Also we shall compute the effect of the δ -Serre operators on f^\sharp ; cf. Propositions 8.3 and 8.4.

4) It would be interesting to extend the above Theorem to f 's of higher weight.

5) Let $g = 1$, $f^\sharp := f_1^\sharp$, and $0 \leq a < b \leq r$, $r \geq 2$; then the δ -eigenforms $(f^a)^{\phi^{b-a}} \cdot f^\sharp$ and $f^a \cdot f^b \cdot f^\rho$ of order r have the same weight $-\phi^a - \phi^b$ and belong to f with the same character and same exponent $e = 1$. One can ask if these forms are R_p -linearly dependent. The answer is negative as one will see in Theorem 7.1.

3. REVIEW OF EICHLER-SHIMURA AND MANIN-DRINFELD

We need to review some basic facts about modular curves. The references for this section are [17, 10, 20, 8]. Fix an integer $N \geq 4$. Then the *modular curve*

$$(3.1) \quad Y_1(N)/\mathbf{Z}[1/N]$$

is the scheme whose S -points (S any scheme over $\mathbf{Z}[1/N]$) identify with isomorphism classes of pairs (E, α) where E/S is an elliptic curve and $\alpha : (\mathbf{Z}/N\mathbf{Z})_S \rightarrow E$ is a closed immersion of group schemes. Recall that $Y_1(N)$ is smooth affine of relative dimension one over $\mathbf{Z}[1/N]$, with geometrically irreducible fibers.

Let now l be a prime integer not dividing N . There is a scheme

$$(3.2) \quad Y_1(N, l)/\mathbf{Z}[1/Nl]$$

whose S -points identify with triples (E, α, H) where (E, α) is as above and H is a finite flat subgroup scheme of E of rank l . Recall that $Y_1(N, l)$ is smooth affine

of relative dimension one over $\mathbf{Z}[1/Nl]$, with geometrically irreducible fibers. Over $\mathbf{Z}[1/Nl]$ one can consider the natural projections

$$(3.3) \quad \begin{aligned} \sigma_1, \sigma_2 : Y_1(N, l) &\rightarrow Y_1(N) \\ \sigma_1(E, \alpha, H) &= (E, \alpha), \\ \sigma_2(E, \alpha, H) &= (E/H, u \circ \alpha), \end{aligned}$$

where $u : E \rightarrow E/H$ is the canonical projection. Moreover σ_1 and σ_2 are étale above $\mathbf{Z}[1/Nl]$. For details on the discussion above see [17] pp. 87, 117, 125, 129, [10], pp. 69-72, [8], pp. 212-213. (For σ_1 we use the convention in [8] rather than that in [10].)

In what follows we need to consider the “compactified” situation. We assume $N > 4$. Recall that $Y_1(N)$ is an open set in its *Deligne-Rapoport compactification* $X_1(N)/\mathbf{Z}[1/N]$ which is a proper smooth scheme. Cf. [10], p. 79. The complex points of $X_1(N) \setminus Y_1(N)$ are called the *cusps* of $X_1(N)$ and they come from $\mathbf{Z}[1/N, \zeta_N]$ -points of $X_1(N)$. As usual we denote by $J_1(N)$ the Jacobian of $X_1(N)$ viewed as an Abelian scheme over $\mathbf{Z}[1/N]$. Let $X_1(N, l)_{\mathbf{C}}$ be a smooth complete model over \mathbf{C} of the complex curve $Y_1(N, l)_{\mathbf{C}}$; the morphisms in Equation 3.3 induce morphisms

$$\sigma_1, \sigma_2 : X_1(N, l)_{\mathbf{C}} \rightarrow X_1(N)_{\mathbf{C}}.$$

These morphisms induce endomorphisms $T(l)_*$ of $J_1(N)_{\mathbf{C}}$ as follows: if D is a divisor of degree 0 on $X_1(N)_{\mathbf{C}}$ and $[D]$ is the point of $J_1(N)_{\mathbf{C}}$ representing D then

$$(3.4) \quad T(l)_*[D] := [\sigma_2^* \sigma_1^* D].$$

The endomorphisms $T(l)_*$ have models over $\mathbf{Z}[1/N]$ (arising from Néron model theory); cf. [8]. We will need the following basic construction due to Eichler-Shimura (cf. [8], p. 215, [11], pp. 241-242):

Theorem 3.1. (*Eichler-Shimura*) *Let $f \in S_2(\Gamma_1(N), \mathbf{C})$ be a newform. Then there exists an Abelian variety $A = A_{\mathbf{Q}}$ defined over \mathbf{Q} of dimension $[K_f : \mathbf{Q}]$, a ring homomorphism $\iota : \mathcal{O}_f \rightarrow \text{End}(A/\mathbf{Q})$, and a dominant homomorphism $\pi : J_1(N)_{\mathbf{Q}} \rightarrow A$ defined over \mathbf{Q} such that the following hold:*

1) *For all primes l we have a commutative diagram*

$$\begin{array}{ccc} J_1(N)_{\mathbf{Q}} & \xrightarrow{T(l)_*} & J_1(N)_{\mathbf{Q}} \\ \pi \downarrow & & \downarrow \pi \\ A & \xrightarrow{\iota(a_l(f))} & A \end{array}$$

2) *The image of the pull-back map*

$$\pi^* : H^0(A_{\mathbf{C}}, \Omega) \rightarrow H^0(J_1(N)_{\mathbf{C}}, \Omega) \simeq S_2(\Gamma_1(N))_{\mathbf{C}}$$

is the \mathbf{C} -linear span of $\{f^\sigma \mid \sigma : K_f \rightarrow \mathbf{C}\}$.

3) *If $f \in S_2(\Gamma_0(N), \mathbf{C})$, $g = 1$, and p is a sufficiently big prime then a_p equals the trace of the p -power Frobenius on the elliptic curve obtained by reducing A mod p .*

Remark 3.2. If $g = 1$ then A in the above Theorem is an elliptic curve over \mathbf{Q} (which, by condition 2 in the Theorem, is uniquely determined by f up to isogeny); we say that f is of CM type or not of CM type according as A has CM or does not have CM. Assertion 3 in the above Theorem, appropriately reformulated, holds for

$\Gamma_1(N)$ and arbitrary g ; for our purposes here we will not need this more general statement.

On the other hand we will need the following theorem due to Manin and Drinfeld (cf. [15] or [19], p. 62):

Theorem 3.3. (*Manin-Drinfeld*) *If D is a divisor of degree 0 on $X_1(N)_{\mathbf{C}}$ supported in the set of cusps then its image $[D]$ in $J_1(N)_{\mathbf{C}}$ is a torsion point.*

4. δ -CHARACTERS

We start by reviewing some concepts from [5, 3]. A δ -morphism of order r , $f : X \rightarrow Y$, between two R -schemes is a rule that attaches to any prolongation sequence S^* of p -adically complete rings a map $f_{S^*} : X(S^0) \rightarrow Y(S^r)$ which is functorial in S^* . In the special case when X is smooth over R and $Y = \mathbf{A}^1$ is the affine line any δ -morphism $f : X \rightarrow \mathbf{A}^1 = \mathbf{A}_R^1$ is completely determined by the map $f_{R^*} : X(R) \rightarrow \mathbf{A}^1(R) = R$. We denote by $\mathcal{O}^r(X)$ the ring of all δ -morphisms $X \rightarrow \mathbf{A}^1$ of order r . Assume that G is a smooth group scheme over R . A δ -morphism $f : G \rightarrow \mathbf{A}^1 = \mathbf{G}_a$ for which $f_{R^*} : G(R) \rightarrow \mathbf{G}_a(R) = R$ is a group homomorphism into the additive group of R is called a δ -character. We denote by $\mathbf{X}^r(G)$ the R -module of δ -characters of G of order r .

Theorem 4.1. [3] *Let A be an Abelian scheme over R of relative dimension g . Then $(r-1)g \leq \text{rank}_R \mathbf{X}^r(A) \leq rg$.*

Proof. This is contained in [3], pp. 325-326. \square

As explained in [3], the δ -characters $\psi : A \rightarrow \mathbf{G}_a$ of an Abelian variety should be viewed as arithmetic analogues of the Manin maps in [14]; Manin maps are homomorphisms $A(L) \rightarrow L$ defined for any Abelian variety A over a field L of characteristic zero equipped with a non-zero derivation $D : L \rightarrow L$. In our theory the role of the derivation D is played by the p -derivation δ .

Remark 4.2. For $N \geq 4$ the description of points of $Y_1(N)$ immediately implies that we have an identification

$$(4.1) \quad M^r(\Gamma_1(N), R, 0) \simeq \mathcal{O}^r(Y_1(N)_R).$$

More generally, the spaces $M^r(\Gamma_1(N), R, w)$ and $M_{ord}^r(\Gamma_1(N), R, w)$ identify with the spaces $M_{Y_1(N)_R}^r(w)$ and $M_{Y_1(N)_R, ord}^r(w)$ in [6], p. 251, respectively.

The above suggests the following:

Definition 4.3. A δ -modular form $f \in M^r(\Gamma_1(N), R, 0) \simeq \mathcal{O}^r(Y_1(N)_R)$ is called δ -holomorphic if it lies in the image of $\mathcal{O}^r(X_1(N)_R) \rightarrow \mathcal{O}^r(Y_1(N)_R)$. If $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R$ is an embedding then the *cusps* of $X_1(N)_R$ (with respect to ρ) are the R -points of $X_1(N)_R$ obtained as images, via ρ , of the cusps of $X_1(N)_{\mathbf{C}}$ (viewed as $\mathbf{Z}[1/N, \zeta_N]$ -points). A δ -modular form $f \in M^r(\Gamma_1(N), R, 0)$ is a δ -cusp form (with respect to ρ) if it is δ -holomorphic and it vanished at all the cusps of $X_1(N)_R$ (with respect to ρ).

Remark 4.4. For the next Proposition and its proof it is useful to introduce some terminology and record some facts about it. If K is a field, V is an n -dimensional K -linear space, and $T \in \text{End}(V)$ then, by the *eigenvalues* of T on V we mean

the eigenvalues (in an algebraic closure K^a of K) of any matrix in $Mat_n(K)$ representing T . If $W \subset V$ is a subspace with $TW \subset W$ then all eigenvalues of T on W and all eigenvalues of T on V/W are also eigenvalues of T on V . If L is a field extension of K we say that $T \in End(V)$ is *diagonalizable* over L if $V \otimes_K L$ has an L -basis consisting of eigenvectors of T . More generally, $T_1, \dots, T_s \in End(V)$ are said to be *simultaneously diagonalizable* over L if $V \otimes_K L$ has an L -basis consisting of common eigenvectors of T_1, \dots, T_s . We will use the following trivial facts:

- 1) T is diagonalizable over K^a if and only if the minimal polynomial of T on $V \otimes_K K^a$ has simple roots only.
- 2) If T is diagonalizable over K^a and all its eigenvalues are in K then T is diagonalizable over K .
- 3) If T_1, \dots, T_s are pairwise commuting and each of them is diagonalizable over K then T_1, \dots, T_s are simultaneously diagonalizable over K .

Proposition 4.5. *Let A/R be an Abelian scheme of relative dimension g and $\tau_1, \dots, \tau_s \in End(A/R)$ be commuting endomorphisms each of which annihilates a polynomial with \mathbf{Z} -coefficients with simple complex roots only. Let \mathcal{T} be the subring of $End(A/R)$ generated by τ_1, \dots, τ_s and let K be the fraction field of R . Assume the tangent maps $d\tau_1, \dots, d\tau_s \in End(Lie(A_K/K))$ have all their eigenvalues in K . (Here we view $Lie(A_K/K)$ as the tangent space to A_K at the origin.) Then there exist R -linearly independent δ -characters $\psi_1, \dots, \psi_g : A \rightarrow \mathbf{G}_a$ of order 2 and ring homomorphisms $\chi^1, \dots, \chi^g : \mathcal{T} \rightarrow R$ such that $\psi_j \circ \tau = \chi_j(\tau) \cdot \psi_j$ for all $\tau \in \mathcal{T}$, $j = 1, \dots, g$.*

Proof. Let the formal group A^{for} of A be identified with $Spf R[[x]]$ where $x = \{x_1, \dots, x_g\}$ are some variables. Let $L = (L_1, \dots, L_g) \in K[[x]]^g$ be the logarithm of the formal group law of A with respect to x . Let x' and x'' be additional g -tuples of variables and let $K[[x]] \xrightarrow{\phi} K[[x, x']] \xrightarrow{\phi} K[[x, x', x'']]$ be the ring homomorphisms, extending $\phi : R \rightarrow R$, defined by $\phi(x) = x^p + px'$, $\phi(x') = (x')^p + px''$. By Theorem 4.1 we have $rank_R \mathbf{X}^2(A) \geq g$. Note that $End(A/R)^{op}$ acts on $\mathbf{X}^2(A)$ (and hence on $\mathbf{X}^2(A) \otimes K$) by the formula $(\tau, \psi) \mapsto \tau^* \psi := \psi \circ \tau$, $\tau \in End(A/R)$. On the other hand $End(A/R)^{op}$ naturally acts on $R[[x]]$ via $(\tau, x) \mapsto \tau^* x$ and one can extend this action uniquely to an action on $K[[x, x', x'']] = K[[x, \phi(x), \phi^2(x)]]$ such that the induced endomorphisms τ^* on the latter ring satisfy $\phi(\tau^* G) = \tau^*(\phi(G))$ for all $G \in K[[x, x']]$. By [3], Lemma 2.8, there exists an injective K -linear $End(A/R)^{op}$ -equivariant map $\mathbf{X}^2(A) \otimes K \rightarrow K[[x, x', x'']]$ whose image lies in the K -linear space

$$(4.2) \quad V := \sum_{i=1}^g (K \cdot L_i + K \cdot \phi(L_i) + K \cdot \phi^2(L_i)).$$

The logarithm $L : A_K^{for} \rightarrow (\mathbf{G}_{a,K}^{for})^g$ is an isomorphism of formal groups over K hence, in particular, the endomorphisms $\tau_i : A^{for} \rightarrow A^{for}$ induce, via L , endomorphisms of $(\mathbf{G}_{a,K}^{for})^g$, hence matrices $M(\tau_i) = (m_{js}(\tau_i)) \in Mat_g(K)$ satisfying

$$(4.3) \quad \tau_i^* L_j = \sum_{s=1}^g m_{js}(\tau_i) L_s.$$

On the other hand, taking the *Lie* functor we get commutative diagrams

$$\begin{array}{ccccc} \mathrm{Lie}(A_K/K) & \simeq & \mathrm{Lie}(A_K^{for}) & \xrightarrow{dL} & \mathrm{Lie}(\mathbf{G}_{a,K}^{for})^g \\ d\tau_i \downarrow & & d\tau_i \downarrow & & \downarrow M(\tau_i) \\ \mathrm{Lie}(A_K/K) & \simeq & \mathrm{Lie}(A_K^{for}) & \xrightarrow{dL} & \mathrm{Lie}(\mathbf{G}_{a,K}^{for})^g \end{array}$$

showing that the endomorphisms $d\tau_i$ on $\mathrm{Lie}(A_K/K)$ can be represented by the matrices $M(\tau_i)$. Since all eigenvalues of $d\tau_i$ on $\mathrm{Lie}(A_K/K)$ lie in K the same is true about the eigenvalues of the matrices $M(\tau_i)$. On the other hand Equation 4.3 implies that

$$\tau_i^*(\phi^e(L_j)) = \sum_{s=1}^g \phi^e(m_{js}(\tau_i))\phi^e(L_s)$$

for $e = 1, 2$. Hence τ_i act on the space V in Equation 4.2 via a matrix of the form

$$\begin{bmatrix} M(\tau_i) & 0 & 0 \\ 0 & \phi M(\tau_i) & 0 \\ 0 & 0 & \phi^2 M(\tau_i) \end{bmatrix}$$

The above matrices have all their eigenvalues in K . We deduce that all the eigenvalues of τ_i on $\mathbf{X}^2(A) \otimes K$ are in K . On the other hand, since each τ_i annihilates a polynomial with \mathbf{Z} -coefficients having simple complex roots only, the same will be true about τ_i acting on V hence the minimal polynomial of τ_i on $V \otimes_K K^a$ has simple roots only so the minimal polynomial of τ_i acting on $\mathbf{X}^2(A) \otimes K^a$ has simple roots only. So τ_i acting on $\mathbf{X}^2(A) \otimes K$ is diagonalizable over K^a (by Remark 4.4, assertion 1) hence over K (by Remark 4.4, assertion 2). Then, by Remark 4.4, assertion 3, τ_1, \dots, τ_s acting on $\mathbf{X}^2(A) \otimes K$ are simultaneously diagonalizable over K . So there exist at least g K -linearly independent δ -characters $\psi_1, \dots, \psi_g \in \mathbf{X}^2(A) \otimes K$ such that $\tau_i^* \psi_j = \lambda_{ij} \cdot \psi_j$, $\lambda_{ij} \in K$. Multiplying ψ_j by a power of p we may assume $\psi_j \in \mathbf{X}^2(A)$. Now, since $\mathrm{End}(A/R) \subset \mathrm{Mat}_{2g}(\mathbf{Z})$, τ_i are integral over \mathbf{Z} hence so are the λ_{ij} 's. Since R is integrally closed, $\lambda_{ij} \in R$. Clearly then, for $\tau \in \mathcal{T}$, we have $\tau^* \psi_j = \chi^j(\tau) \cdot \psi_j$ for some $\chi^j(\tau) \in R$ and χ^j defines a ring homomorphism $\mathcal{T} \rightarrow R$. \square

5. PROOF OF THEOREM 2.5

Fix a cusp P^0 of $X_1(N)_{\mathbf{C}}$ and consider the morphisms

$$(5.1) \quad X_1(N)_{\mathbf{C}} \xrightarrow{\beta} J_1(N)_{\mathbf{C}} \xrightarrow{\pi} A_{\mathbf{C}},$$

where β is the Abel-Jacobi map sending $P^0 \mapsto 0$ and A is as in Theorem 3.1. Let $M \in \mathbf{Z}$ be divisible by N such that the embedding $\iota : \mathcal{O}_f \rightarrow \mathrm{End}(A)$, all the cusps of $X_1(N)_{\mathbf{C}}$ and all the objects and morphisms in Equation 5.1 have (compatible) models over $\mathbf{Z}[1/M, \zeta_N]$.

Let $p \in \mathbf{Z}$ be any prime not dividing M and unramified in $F := \tilde{K}_f(\zeta_N)$ where \tilde{K}_f is the normal closure of K_f in \mathbf{C} . Assume we are given an embedding $\rho : \mathcal{O}_f[1/N, \zeta_N] \rightarrow R_p = \hat{\mathbf{Z}}_p^{ur}$. The latter can be lifted to an embedding $\tilde{\rho} : \mathcal{O}_F[1/M] \rightarrow R_p$ where \mathcal{O}_F is the ring of integers of F . Then the morphisms in Equation 5.1 induce (by base change via $\tilde{\rho}$) morphisms of R_p -schemes

$$(5.2) \quad X_1(N)_{R_p} \xrightarrow{\beta} J_1(N)_{R_p} \xrightarrow{\pi} A_{R_p}.$$

Select primes l_1, \dots, l_s not dividing Np such that the endomorphisms

$$T(l_1)_*, \dots, T(l_s)_*$$

of $J_1(N)/\mathbf{Z}[1/N]$ generate the subring

$$\mathbf{Z}[T(l')_* \mid l' \text{ prime not dividing } Np]$$

of $\text{End}(J_1(N)/\mathbf{Z}[1/N])$. Let $\mathcal{T}^{(Np)}$ be the subring of $\text{End}(A_{R_p}/R_p)$ generated by $\iota(a_{l'})$ with l' prime not dividing Np . Clearly $\mathcal{T}^{(Np)}$ is generated as a ring by $\iota(a_{l_1}), \dots, \iota(a_{l_s})$.

Claim. *Each of the maps $d(\iota(a_{l_1})), \dots, d(\iota(a_{l_s})) \in \text{End}(\text{Lie}(A_{K_p}/K_p))$ has all its eigenvalues in K_p , the fraction field of R_p .*

Indeed an eigenvalue of $d(\iota(a_{l_i}))$ on $\text{Lie}(A_{K_p}/K_p)$ is also an eigenvalue of $d(\iota(a_{l_i}))$ on $\text{Lie}(A_{\mathbf{Q}}/\mathbf{Q})$ hence on $H^0(A_{\mathbf{C}}, \Omega)$. By the Eichler-Simura Theorem 3.1 the latter identifies with the \mathbf{C} -linear span V_f of $\{f^\sigma \mid \sigma : K_f \rightarrow \mathbf{C}\}$ and the action of $d(\iota(a_{l_i}))$ corresponds to the action on V_f of $T_2(l_i)$. But the eigenvalues of $T_2(l_i)$ on V_f are clearly in \tilde{K}_f and our Claim is proved.

Now each a_{l_i} , being an element of the field K_f , annihilates a polynomial with \mathbf{Z} -coefficients having simple complex roots only. Hence the same is true about $\iota(a_{l_i})$. By Proposition 4.5 there exist $g = [K_f : \mathbf{Q}]$ R_p -linearly independent δ -characters of order 2,

$$(5.3) \quad \psi_1, \dots, \psi_g : A_{R_p} \rightarrow \mathbf{G}_{a, R_p} = \mathbf{A}_{R_p}^1$$

and ring homomorphisms $\chi^1, \dots, \chi^g : \mathcal{T}^{(Np)} \rightarrow R_p$ such that for any prime l' not dividing Np we have

$$\psi_j \circ \iota(a_{l'}) = \chi^j(\iota(a_{l'})) \cdot \psi_j, \quad j = 1, \dots, g.$$

We may then consider the δ -morphisms of order 2,

$$(5.4) \quad f_j^\# := \psi_j \circ \pi \circ \beta : X_1(N)_{R_p} \rightarrow \mathbf{A}_{R_p}^1, \quad j = 1, \dots, g.$$

Their restrictions to $Y_1(N)_{R_p}$ can be viewed as δ -modular forms of weight 0.

Let now F' be a number field containing F , let v' be a valuation on F' above p and let P be any $\mathcal{O}_{F', v'}$ -point of $Y_1(N)$ where $\mathcal{O}_{F', v'}$ is the valuation ring of F' at v' . Since $\mathcal{O}_F[1/M] \subset \mathcal{O}_{F', v'}$ all cusps of $X_1(N)$ are $\mathcal{O}_{F', v'}$ -points and $T(l)_*$ has a model over $\mathcal{O}_{F', v'}$. So one may write

$$\sigma_{2*} \sigma_1^* P = \sum_{i=1}^{l+1} P_i, \quad \sigma_{2*} \sigma_1^* P^0 = \sum_{i=1}^{l+1} P_i^0,$$

where P_i are distinct $\mathcal{O}_{F'', v''}$ -points of $X_1(N)$, $\mathcal{O}_{F'', v''}$ an unramified finite extension of $\mathcal{O}_{F', v'}$, and P_i^0 are (a priori not necessarily distinct) cusps of $X_1(N)$ so they are $\mathcal{O}_{F', v'}$ -points. So, for l prime, not dividing Np , Equation 3.4 yields

$$(5.5) \quad T(l)_*(\beta(P)) = \sum_{i=1}^{l+1} \beta(P_i) - \sum_{i=1}^{l+1} \beta(P_i^0).$$

Choose now an embedding $\mathcal{O}_{F'', v''} \rightarrow R_p$ extending the fixed embedding $\mathcal{O}_F[1/M] \rightarrow R_p$ above; in particular the points P_i and P_i^0 can be viewed as R_p -points of $X_1(N)$.

By Theorem 3.3, $\beta(P_i^0)$ are torsion points hence their image via $\psi_j \circ \pi$ is 0. Using this and the fact that ψ_j and π are homomorphisms, we get, by Equation 5.5:

$$\begin{aligned}
 \psi_j \pi T(l)_*(\beta(P)) &= \psi_j \pi (\sum_{i=1}^{l+1} \beta(P_i) - \sum_{i=1}^{l+1} \beta(P_i^0)) \\
 (5.6) \qquad \qquad \qquad &= \sum_{i=1}^{l+1} f_j^\#(P_i) \\
 &= (T(l)f_j^\#)(P).
 \end{aligned}$$

On the other hand, using Theorem 3.1, we get

$$(5.7) \qquad \psi_j \pi T(l)_*(\beta(P)) = \psi_j (\iota(a_l) \pi \beta(P)) = \chi^j(\iota(a_l)) \cdot f_j^\#(P).$$

Equations 5.7 and 5.6 imply that

$$(5.8) \qquad (T(l)f_j^\#)(P) = \chi^j(\iota(a_l)) \cdot f_j^\#(P).$$

In the above equality P has coordinates in $\mathcal{O}_{F',v'}$. Since F' and v' are arbitrary, and since, by smoothness, the image of

$$\bigcup_{F',v'} Y_1(N)(\mathcal{O}_{F',v'}) \rightarrow Y_1(N)(R_p)$$

is p -adically dense in $Y_1(N)(R_p)$ it follows, by continuity, that Equation 5.8 holds for any R_p -point P of $Y_1(N)$. This implies that $T(l)f_j^\# = \chi_j(a_l) \cdot f_j^\#$ where $\chi_j : \mathcal{O}_f^{(Np)} \rightarrow R_p$ is the composition

$$\mathcal{O}_f^{(Np)} \xrightarrow{\iota} \mathcal{T}^{(Np)} \xrightarrow{\chi^j} R_p.$$

To conclude it is enough to check that $f_1^\#, \dots, f_g^\#$ are R_p -linearly independent. Assume $\sum c_j f_j^\# = 0$, $c_j \in R_p$. For large n , the image of the natural map

$$X_1(N)(R_p)^n \rightarrow J_1(N)(R_p)$$

contains all the R -points of an open subset U of $J_1(N)$. It follows that the restriction of $(\sum c_j \psi_j) \circ \pi$ to $U(R_p)$ is 0. This immediately implies that $(\sum c_j \psi_j) \circ \pi = 0$. Hence $\sum c_j \psi_j = 0$ which implies $c_j = 0$ for all j . This ends our proof.

Remark 5.1. Note that the δ -modular forms $f_j^\#$ constructed above are δ -cusp forms (with respect to ρ) in the sense of Definition 4.3. Indeed $f_j^\#$ come from δ -morphisms

$$\psi \circ \pi \circ \beta : X_1(N)_{R_p} \rightarrow \mathbf{A}_{R_p}^1,$$

so they are δ -holomorphic. Now, by the Manin-Drinfeld Theorem 3.3, the images of the cusps in $J_1(N)(R)$ via the Abel-Jacobi map

$$\beta : X_1(N)(R_p) \rightarrow J_1(N)(R_p)$$

are torsion points. On the other hand

$$\psi \circ \pi : J_1(N)(R_p) \rightarrow \mathbf{G}_{a,R_p}(R_p) = R_p$$

is a homomorphism with torsion free target so it vanishes on torsion points.

6. δ -FOURIER EXPANSIONS

We start by recalling the background of *classical* Fourier expansions; cf. [10], p. 112. (The discussion in loc. cit. involves the model $X_\mu(N)$ instead of the model $X_1(N)$ used here but the two models, and hence the two theories, are isomorphic over $\mathbf{Z}[1/N, \zeta_N]$ cf. [10], p. 113.) There is a point $s_\infty : \mathbf{Z}[1/N, \zeta_N] \rightarrow X_1(N)_{\mathbf{Z}[1/N, \zeta_N]}$ arising from the generalized elliptic curve $\mathbf{P}_{\mathbf{Z}[1/N, \zeta_N]}^1$ with its canonical embedding of $\mu_{N, \mathbf{Z}[1/N, \zeta_N]} \simeq (\mathbf{Z}/N\mathbf{Z})_{\mathbf{Z}[1/N, \zeta_N]}$; the complex point corresponding to s_∞ is the cusp $\Gamma_1(N) \cdot \infty$. The map s_∞ is a closed immersion; denote by $\tilde{X}_1(N)_{\mathbf{Z}[1/N, \zeta_N]}$ the completion of $X_1(N)_{\mathbf{Z}[1/N, \zeta_N]}$ along the image of s_∞ . Now consider the Tate generalized elliptic curve

$$\text{Tate}(q)/\mathbf{Z}[1/N, \zeta_N][[q]];$$

it has a canonical immersion α_{can} of $\mu_{N, \mathbf{Z}[1/N, \zeta_N]} \simeq (\mathbf{Z}/N\mathbf{Z})_{\mathbf{Z}[1/N, \zeta_N]}$ so there is an induced map $\text{Spec } \mathbf{Z}[1/N, \zeta_N] \rightarrow X_1(N)_{\mathbf{Z}[1/N, \zeta_N]}$ (which further composed with $\text{Spec } \mathbf{Z}[1/N, \zeta_N] \rightarrow \text{Spec } \mathbf{Z}[1/N, \zeta_N][[q]]$, $q \mapsto 0$, equals s_∞). There is an induced isomorphism

$$(6.1) \quad \text{Spf } \mathbf{Z}[1/N, \zeta_N][[q]] \rightarrow \tilde{X}_1(N)_{\mathbf{Z}[1/N, \zeta_N]}.$$

There is a canonical 1-form ω_{can} on the elliptic curve $\text{Tate}(q)/\mathbf{Z}[1/N, \zeta_N]((q))$ over $\mathbf{Z}[1/N, \zeta_N]((q)) := \mathbf{Z}[1/N, \zeta_N][[q]][1/q]$ such that the induced map

$$S_2(\Gamma_1(N), \mathbf{C}) \rightarrow \mathbf{C}((q)), \quad f \mapsto f_\infty := f_\infty(q) := f(\text{Tate}(q)/\mathbf{C}((q)), \alpha_{can}, \omega_{can}),$$

has image in $q\mathbf{C}[[q]]$ and is the classical Fourier expansion at the cusp $\Gamma_1(N) \cdot \infty$. (Here we interpret f as a function of triples in the style of [16].)

Next we move to the δ -theory. Fix a prime p not dividing N and fix a homomorphism $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$. We may consider the prolongation sequence S_∞^* defined by

$$S_\infty^r := R_p((q))^\wedge[q', q'', \dots, q^{(r)}]^\wedge,$$

where $^\wedge$ means p -adic completion and $q', q'', \dots, q^{(r)}$ are new indeterminates; S_∞^{r+1} is viewed as an S_∞^r -algebra via the inclusion and the p -derivations $\delta : S_\infty^r \rightarrow S_\infty^{r+1}$ are the unique p -derivations satisfying $\delta q^{(i)} = q^{(i+1)}$. Explicitly, they are defined as follows. First extend $\phi : R_p \rightarrow R_p$ to ring homomorphisms $\phi : S_\infty^r \rightarrow S_\infty^{r+1}$ by requiring that

$$(6.2) \quad \phi(q) := q^p + pq', \quad \phi(q') := (q')^p + pq'', \dots$$

and then define $\delta : S_\infty^r \rightarrow S_\infty^{r+1}$ by

$$(6.3) \quad \delta F := \frac{\phi(F) - F^p}{p}.$$

Finally define the δ -Fourier expansion map

$$E_{\infty, \rho} : M^r(\Gamma_1(N), R_p, w) \rightarrow S_\infty^r, \quad f \mapsto E_{\infty, \rho}(f) = f_{\infty, \rho},$$

by the formula

$$f_{\infty, \rho} := f_{\infty, \rho}(q, q', \dots, q^{(r)}) := f(\text{Tate}(q)/S_{for}^0, \alpha_{can}, \omega_{can}, S_\infty^*).$$

Note that if $f \in S_m(\Gamma_1(N), \mathbf{C})$ is a newform and $\rho : \mathcal{O}_f[1/N, \zeta_N] \rightarrow R_p$ extends the homomorphism $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ above then, by functoriality, $(f^\rho)_{\infty, \rho}(q) = (f_\infty(q))^\rho$, where $(f^\rho)_{\infty, \rho}(q)$ is the δ -Fourier expansion of $f^\rho \in M^0(\Gamma_1(N), R_p, m)$ and $(f_\infty(q))^\rho$ is the image of the classical Fourier expansion $f_\infty(q) \in \mathcal{O}_f[[q]]$ via the

natural map $\rho : \mathcal{O}_f[[q]] \rightarrow R_p[[q]]$. If $\mathcal{O}_f = \mathbf{Z}$ we simply have $(f_\infty(q))^\rho = f_\infty(q)$. Also note that the δ -Fourier expansion map commutes with ϕ i.e. $(f^\phi)_{\infty,\rho} = (f_{\infty,\rho})^\phi$.

Lemma 6.1. (*δ -expansion principle*) *The δ -Fourier expansion map*

$$E_{\infty,\rho} : M^r(\Gamma_1(N), R_p, w) \rightarrow S_\infty^r$$

is injective, with torsion free cokernel.

Proof. This follows from [6], Proposition 4.43, by “taking fractions” with denominators powers of a local equation defining the cusp ∞ . \square

For two elements $F, G \in S_\infty^r$ we write $F \sim G$ if $F = \lambda \cdot G$ with $\lambda \in R_p^\times$ and we write $\bar{F} \sim \bar{G}$ if the images,

$$\bar{F}, \bar{G} \in S_\infty^r \otimes k = k((q))[q', q'', \dots, q^{(r)}]$$

of F and G satisfy $\bar{F} = c \cdot \bar{G}$ for some $c \in k^\times$, $k := R_p/pR_p$. Let

$$\Psi = \Psi(q, q') := \frac{1}{p} \cdot \log \left(1 + p \frac{q'}{q^p} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} p^{n-1} n^{-1} \left(\frac{q'}{q^p} \right)^n \in R_p((q))^\wedge [q']^\wedge.$$

Then the δ -Fourier expansion of $f^r = f_{crys}^r \in M^r(\Gamma_1(N), R_p, -1 - \phi^r)$ is given by:

Lemma 6.2. [1]

$$f_{\infty,\rho}^r \sim \Psi^{\phi^{r-1}} + p\Psi^{\phi^{r-2}} + \dots + p^{r-1}\Psi \in R_p((q))^\wedge [q', \dots, q^{(r)}]^\wedge.$$

In particular

$$\overline{f_{\infty,\rho}^r} \sim \left(\frac{q'}{q^p} \right)^{p^{r-1}} \in k((q))[q', \dots, q^{(r)}].$$

Next we would like to compute the δ -Fourier expansion of the forms $f^\sharp = f_1^\sharp$ in Theorem 2.5 for $g = 1$.

Theorem 6.3. *Let $f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{C})$ be a newform with $K_f = \mathbf{Q}$ which is not of CM type. Then, for any sufficiently large prime p and any embedding $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ there is a unique δ -eigenform $f^\sharp \in M^2(\Gamma_1(N), R_p, 0)$ with δ -Fourier expansion:*

$$(6.4) \quad f_{\infty,\rho}^\sharp(q, q', q'') = \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi^2} - a_p q^{n\phi} + p q^n) \in R_p((q))^\wedge [q', q'']^\wedge.$$

Moreover f^\sharp is a δ -cusp form (with respect to ρ) belonging to f with exponent 0 and

$$(6.5) \quad f_{\infty,\rho}^\sharp(q, 0, 0) = \sum_{(n,p)=1} \frac{a_n}{n} q^n \in R_p[[q]].$$

Remark 6.4. 1) Explicitly, in Equation 6.4 we have:

$$\begin{aligned} q^{n\phi} &= (q^p + p q')^n, \\ q^{n\phi^2} &= [(q^p + p q')^p + p(q')^p + p^2 q'']^n. \end{aligned}$$

2) Uniqueness of f^\sharp follows, of course, from the δ -expansion principle (Lemma 6.1).

3) The series

$$(6.6) \quad \sum_{(n,p)=1} \frac{a_n}{n} q^n$$

is normalized and has coefficients in $\mathbf{Z}_{(p)}$ but not all its coefficients are in \mathbf{Z} . Indeed if the latter were the case then any prime $l \neq p$ would divide a_l . Since, for big enough l , a_l are the traces of Frobenius of an elliptic curve A over \mathbf{Q} taken modulo l it would follow that A has supersingular reduction for sufficiently big l , a contradiction. In particular the series 6.6 is not a (classical) eigenform.

Proof. We begin with a preparatory discussion; in this discussion we will not assume yet that f is not of CM type (for we will use this discussion later in case f is of CM type). We place ourselves in the context of the proof of Theorem 2.5. Since $K_f = \mathbf{Q}$ the field F in that proof equals $\mathbf{Q}(\zeta_N)$. Also we let $\beta : X_1(N)_{\mathbf{C}} \rightarrow J_1(N)_{\mathbf{C}}$ be the Abel-Jacobi map that sends the cusp $P^0 = \infty$ into 0. One can choose A in Theorem 3.1, and hence in the proof of Theorem 2.5, such that

$$\Phi := \pi \circ \beta : X_1(N)_{\mathbf{C}} \rightarrow A_{\mathbf{C}}$$

satisfies

$$\Phi^* \omega_A = c \cdot 2\pi i \cdot f(z) dz = c \cdot \sum_{n \geq 1} a_n q^{n-1} dq,$$

where ω_A is a 1-form on A over \mathbf{Q} , $q = e^{2\pi i z}$ and $c \in \mathbf{Q}^\times$. Cf. [9], p. 19. Let T be an étale coordinate around the origin 0 of A such that T vanishes at 0. Let $L(T) \in \mathbf{Q}[[T]]$ be the logarithm of the formal group of A associated to T ; cf. [22]. Then $\omega_A = \omega(T) dT \in \mathbf{Q}[[T]] \cdot dT$; replacing ω_A by a \mathbf{Q}^\times -multiple of it we may (and will) assume $\omega(0) = c$. Let $m \in \mathbf{Z}$ be such that $X_1(N)$, A , T , ω_A have (compatible) models over $\mathbf{Z}[1/m]$. Then P^0 and Φ are defined over $\mathcal{O} := \mathbf{Z}[1/Nm, \zeta_N]$. We have an induced homomorphism $\Phi^* : \mathcal{O}[[T]] \rightarrow \mathcal{O}[[q]]$ and we set $\Phi^*(T) = \varphi(q) \in \mathcal{O}[[q]]$. Since $\omega_A = c \cdot \frac{dL}{dT} \cdot dT$ we get

$$c \cdot \left(\sum_{n \geq 1} a_n q^{n-1} \right) dq = \Phi^* \omega_A = c \cdot \frac{dL}{dT}(\varphi(q)) \cdot \frac{d\varphi}{dq}(q) \cdot dq = c \cdot \frac{d}{dq}(L(\varphi(q))).$$

Setting $q = 0$ in the coefficients of dq we get that $\frac{d\varphi}{dq}(0) = 1$. Also we deduce that

$$L(\varphi(q)) = \sum_{n \geq 1} \frac{a_n}{n} q^n.$$

Now, for p sufficiently large, $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ induces a homomorphism $\rho : \mathbf{Z}[1/Nm, \zeta_N] \rightarrow R_p$.

From this point on we assume f is not of CM type. Since A is not a CM elliptic curve it follows that A_{R_p} does not have a lift of Frobenius (i.e. there is no morphism of schemes $\tilde{\phi} : A \rightarrow A$ over \mathbf{Z} lifting the morphism $\text{Spec } R_p \rightarrow \text{Spec } R_p$ induced by ϕ such that the reduction mod p of $\tilde{\phi}$ is the p -power Frobenius on $A_{R_p} \otimes k$.) Since A_{R_p} does not have a lift of Frobenius, by [6], Theorem 7.22 and [4], Theorem 1.10, one can assume the δ -character ψ in the proof of Theorem 2.5 gives rise to the series (still denoted by)

$$\psi = \frac{1}{p}(\phi^2 - a_p \phi + p)L(T) \in R_p[[T]][T', T'']^\wedge,$$

where ϕ is viewed here as naturally extended to series with K -coefficients. Then the δ -Fourier expansion of the form $f^\sharp := f_1^\sharp$ provided by the proof of Theorem 2.5 equals

$$\begin{aligned}
 f_{\infty, \rho}^\sharp &= \Phi^* \psi = \Phi^* \left(\frac{1}{p} (\phi^2 - a_p \phi + p) L(T) \right) \\
 &= \frac{1}{p} \{ (\phi^2 - a_p \phi + p) (L(T)) \} (\varphi(q), \delta(\varphi(q)), \delta^2(\varphi(q))) \\
 (6.7) \quad &= \frac{1}{p} (\phi^2 - a_p \phi + p) (L(\varphi(q))) = \frac{1}{p} (\phi^2 - a_p \phi + p) \left(\sum_{n \geq 1} \frac{a_n}{n} q^n \right) \\
 &= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi^2} - a_p q^{n\phi} + p q^n)
 \end{aligned}$$

and our Equation 6.4 follows.

Setting $q' = q'' = 0$ in this equation we get

$$\begin{aligned}
 f_{\infty, \rho}^\sharp(q, 0, 0) &= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{np^2} - a_p q^{np} + p q^n) = \sum_{c \geq 1} c_n q^n, \\
 c_n &= \frac{1}{p} \left(\frac{a_n/p^2}{n/p^2} - \frac{a_n/p a_p}{n/p} + \frac{a_n}{n} p \right).
 \end{aligned}$$

Here, by definition, $a_x = 0$ for $x \in \mathbf{Q} \setminus \mathbf{Z}$. Note that for p not dividing n we get $c_n = a_n/n$. For $n = pm$ with p not dividing m we get

$$c_n = \frac{a_n}{n} - \frac{a_m a_p}{pm} = 0.$$

For $n = p^i m$ with p not dividing m and $i \geq 2$ we get, by Equation 2.2, that

$$c_n = \frac{a_{p^{i-2}m}}{p^{i-1}m} - \frac{a_{p^{i-1}m} a_p}{n} + \frac{a_n}{n} = \frac{a_m}{n} (p a_{p^{i-2}} - a_{p^{i-1}} a_p + a_{p^i}) = 0.$$

Equation 6.5 follows. \square

In the CM case we have the following:

Theorem 6.5. *Let $f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{C})$ be a newform with $K_f = \mathbf{Q}$ which is of CM type (corresponding to an imaginary quadratic field \mathcal{K}). Then, for any sufficiently large prime p and any embedding $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ there is a unique δ -eigenform $f^\sharp \in M^2(\Gamma_1(N), R_p, 0)$ such that the following hold:*

- 1) *If p does not split in \mathcal{K} then Equation 6.4 holds (with $a_p = 0$).*
- 2) *If p splits in \mathcal{K} and if pu is the unique root in $p\mathbf{Z}_p^\times$ of the polynomial $x^2 - a_p x + p$ then the following equation holds:*

$$(6.8) \quad f_{\infty, \rho}^\sharp(q, q', q'') = \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi} - pu q^n) \in R_p((q))^\wedge [q', q'']^\wedge.$$

Moreover f^\sharp is a δ -cusp form (with respect to ρ), f^\sharp belongs to f with exponent 0, and the following hold:

- 1') *If p does not split in \mathcal{K} then Equation 6.5 holds;*
- 2') *If p splits in \mathcal{K} then*

$$(6.9) \quad f_{\infty, \rho}^\sharp(q, 0, 0) = -u \cdot \sum_{(m, p)=1} \frac{a_m}{m} \sum_{i \geq 0} u^i q^{mp^i} \in R_p[[q]].$$

Proof. Let us go back to the preparatory discussion in the proof of Theorem 6.3. In our case here A is an elliptic curve over \mathbf{Q} with CM by an order of \mathcal{K} . By [23], p. 180, there exists an elliptic curve A' over \mathbf{Q} and an isogeny defined over \mathbf{Q} , $A \rightarrow A'$, such that A' has CM by the ring of integers $\mathcal{O}_{\mathcal{K}}$ of \mathcal{K} . Replacing A by A' we may assume that A itself has CM by $\mathcal{O}_{\mathcal{K}}$. Since A is defined over \mathbf{Q} , \mathcal{K} has class number one; cf. [23], pp. 118, 121. By standard facts about elliptic curves with CM ([23], pp 184, 133) we have that for p large enough the following hold:

I) If p does not split in \mathcal{K} then A_{R_p} has supersingular reduction, $a_p = 0$, and A_{R_p} doesn't have a lift of Frobenius.

II) If p splits in \mathcal{K} then A_{R_p} has ordinary reduction, $a_p \not\equiv 0 \pmod{p}$, and A_{R_p} has a lift of Frobenius.

If we are in case I the argument in the proof of Theorem 6.3 applies. Assume we are in case II. Then, by [4], Theorem 1.10, ψ gives rise to the series (still denoted by)

$$\psi = \frac{1}{p}(\phi - up)L(T) \in R_p[[T]][T']^{\wedge}.$$

Then the δ -Fourier expansion of the form $f^{\#}$ provided by the proof of Theorem 2.5 equals

$$\begin{aligned} f_{\infty, \rho}^{\#} &= \Phi^* \psi \\ &= \Phi^* \left(\frac{1}{p}(\phi - up)L(T) \right) \\ &= \frac{1}{p} \{ (\phi - up)(L(T)) \} (\varphi(q), \delta(\varphi(q))) \\ (6.10) \quad &= \frac{1}{p}(\phi - up)L(\varphi(q)) \\ &= \frac{1}{p}(\phi - up) \left(\sum_{n \geq 1} \frac{a_n}{n} q^n \right) \\ &= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi} - upq^n) \end{aligned}$$

and our Equation 6.8 follows.

Setting $q' = q'' = 0$ in this equation we get

$$\begin{aligned} f_{\infty, \rho}^{\#}(q, 0, 0) &= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi} - upq^n) = \sum_{c \geq 1} c_n q^n, \\ c_n &= \frac{a_{n/p} - ua_n}{n}. \end{aligned}$$

Note that for p not dividing n we get $c_n = -ua_n/n$. For $n = p^i m$ with p not dividing m and $i \geq 1$ we get, by Equation 2.2, that

$$c_n = \frac{a_m}{m} \cdot \frac{a_{p^{i-1}} - ua_{p^i}}{p^i}.$$

We claim that

$$a_{p^{i-1}} - ua_{p^i} = -p^i u^{i+1},$$

and this will, of course, end the proof of the equality in Equation 6.9. To check the claim note that, for $i = 1$, we get $a_1 - ua_p = -pu^2$. In general we proceed by

induction, using Equations 2.2:

$$\begin{aligned}
a_{p^i} - ua_{p^{i+1}} &= a_{p^i} - u(a_{p^i}a_p - pa_{p^{i-1}}) \\
&= (1 - ua_p)a_{p^i} + upa_{p^{i-1}} \\
&= -pu^2a_{p^i} + upa_{p^{i-1}} \\
&= up(a_{p^{i-1}} - ua_{p^i}) \\
&= -p^{i+1}u^{i+2},
\end{aligned}$$

and our claim is proved. \square

7. INDEPENDENCE OF f^\sharp FROM f AND $f^{(-1)}$

Using δ -Fourier expansions it is possible to prove a variant of the result on the independence of f^\sharp from f contained in assertion 2 of Theorem 2.5 and also a result on the independence of f^\sharp from $f^{(-1)}$.

Theorem 7.1. *Let $f \in S_2(\Gamma_0(N), \mathbf{C})$ be a newform of weight 2 on $\Gamma_0(N)$, $N > 4$, with $K_f = \mathbf{Q}$. For any sufficiently large prime p and any embedding $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ let $f^\sharp \in M^2(\Gamma_1(N), R_p, 0)$ be the unique form in Theorems 6.3 or 6.5 according as f is of non-CM or CM type respectively. Then the following hold*

1) *There is no $G \in \mathcal{J}$ such that $G \cdot f^\sharp$ belongs to the J -linear span of*

$$\{f^\rho, f^{\rho\phi}, f^{\rho\phi^2}, f^{\rho\phi^3}, \dots\}.$$

2) *f^\sharp does not belong to the I_{ord} -linear span in $M_{ord}^r(\Gamma_1(N), R_p, 0)$ of*

$$\{f^{(-1)\rho}, f^{(-1)\rho\phi}, f^{(-1)\rho\phi^2}, f^{(-1)\rho\phi^3}, \dots\}.$$

Proof. We prove assertion 1. Since there are infinitely many primes l of ordinary reduction for the elliptic curve A we may choose two such distinct primes l_1 and l_2 ; so $a_{l_1}a_{l_2} \neq 0$. Let p be a prime which is sufficiently big so that Theorem 6.3 holds in case A doesn't have CM (respectively Theorem 6.5 holds in case A has CM) and, in addition, p does not divide $l_1l_2(l_1 - l_2)a_{l_1}a_{l_2}$. Fix a homomorphism $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ and let f^\sharp be the unique form in Theorem 6.3 in case A doesn't have CM (respectively the unique form in Theorem 6.5 in case A has CM).

Assume A doesn't have CM; the case when A has CM is entirely similar and left to reader. Assume there exists $G \in \mathcal{J}$ such that $G \cdot f^\sharp$ is a J -linear combination of forms $f^{\rho\phi^d}$. Hence $G \cdot f^\sharp$ is a R_p -linear combination of forms $F \cdot f^{\rho\phi^d}$ with $F \in \mathcal{J}$. By taking δ -Fourier expansions we get that $G_{\infty, \rho} \cdot f_{\infty, \rho}^\sharp$ is an R_p -linear combination of forms $F_{\infty, \rho} \cdot f_{\infty, \rho}^{\rho\phi^d}$. Reducing modulo p , setting $q'' = 0$, and using Lemma 6.2 and Theorem 6.3 we have a congruence mod p of the form

$$\left(\frac{q'}{q^p}\right)^s \left(\sum_{(m,p)=1} \frac{a_m}{m} q^m + q' B(q, q') \right) \equiv \sum_{j,d \geq 0} \lambda_{dj} \left(\frac{q'}{q^p}\right)^j \left(\sum_{m \geq 1} a_m q^{mp^d} \right)$$

in $R_p((q))[q']^\wedge$, where $s \geq 0$, $\lambda_{ij} \in R_p$, and $B(q, q') \in R_p[[q]][q']$. Let l be either l_1 or l_2 . Identifying the coefficients of $(q')^s q^{l-ps}$ in the above Equation we get

$$\frac{a_l}{l} \equiv \lambda_{0s} \cdot a_l \pmod{p}$$

in R_p . Since $a_l \not\equiv 0 \pmod{p}$ we get

$$1 \equiv l \cdot \lambda_{0s} \pmod{p}.$$

So $l_1 \equiv l_2 \pmod{p}$, a contradiction.

We prove assertion 2. Assume $f^\sharp = \sum_{j \geq 0} \lambda_j f^{(-1)\rho\phi^j}$, $\lambda_j \in I_{ord}$. By Lemma 2.4 we may assume $\lambda_j \in R$. Looking at δ -Fourier expansions we get one of the following equalities:

$$(7.1) \quad \begin{aligned} \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi^2} - a_p q^{n\phi} + p q^n) &= \sum_{j \geq 0} \sum_{(n,p)=1} \lambda_j \frac{a_n}{n} q^{n\phi^j} \\ \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} (q^{n\phi} - p u q^n) &= \sum_{j \geq 0} \sum_{(n,p)=1} \lambda_j \frac{a_n}{n} q^{n\phi^j}. \end{aligned}$$

Setting $q' = q'' = \dots = 0$ and picking out the coefficient of q^{p^j} we get $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = \dots = 0$ in the first case and $\lambda_j = -u^{j+1}$ in the second case respectively. In both situations we clearly get a contradiction. \square

8. δ -SERRE OPERATORS

Recall from [6], p. 255, that the Serre-Katz operators on modular forms [16], p. 169, induce R -derivations

$$\partial_j : M^r(\Gamma_1(N), R, *) \rightarrow M^r(\Gamma_1(N), R, *), \quad j \geq 0,$$

such that if $w = \sum a_i \phi^i$ then

$$\partial_j M^r(\Gamma_1(N), R, w) \subset M^r(\Gamma_1(N), R, w + 2\phi^j).$$

According to [16] (or [6], p. 255) the *Ramanujan form* defines an element

$$P \in M_{ord}^0(\Gamma_1(N), R, 2).$$

(N.B: the P in [16] is 12 times the P in [6]; here we are using the P in [6].) One can consider the R -derivations

$$\partial_j^* : M_{ord}^r(\Gamma_1(N), R, *) := \bigoplus_{w \in W(r)} M_{ord}^r(\Gamma_1(N), R, w) \rightarrow M_{ord}^r(\Gamma_1(N), R, *),$$

where, for $w = \sum a_i \phi^i$, the restriction of ∂_j^* to $M_{ord}^r(\Gamma_1(N), R, w)$ equals

$$\partial_j + a_j p^j P^{\phi^j}.$$

Recall from [6], p. 93, that one also defines $\partial_{**} := \sum_{j \geq 0} p^{-j} \partial_j$. On the other hand one can consider the R -derivation $\theta := q \frac{d}{dq}$ on $S_\infty^0 := R((q))^\wedge$. Exactly as in [6], p. 113, there exist unique R -derivations

$$\theta_j : \bigcup_{r \geq 0} S_\infty^r \rightarrow \bigcup_{r \geq 0} S_\infty^r$$

satisfying the properties

$$(8.1) \quad \begin{aligned} \theta_j \circ \phi^s &= 0, \quad \text{on } S_\infty^0 \quad \text{for } s \neq j, \\ \theta_j \circ \phi^j &= p^j \cdot \phi^j \circ \theta \quad \text{on } S_\infty^0. \end{aligned}$$

Lemma 8.1. *For all $j \geq 0$, $r \geq 0$, $w \in W(r)$, we have an equality of maps*

$$E_{\infty, \rho} \circ \partial_j^* = \theta_j \circ E_{\infty, \rho} : M^r(\Gamma_1(N), R, w) \rightarrow S_\infty^r.$$

In particular we have an equality of maps

$$E_{\infty, \rho} \circ \partial_j = \theta_j \circ E_{\infty, \rho} : M^r(\Gamma_1(N), R, 0) \rightarrow S_\infty^r.$$

Proof. Same argument as in [6], p. 259, where the case of Serre-Tate expansions (rather than Fourier expansions) was considered; to make that argument work one uses [16], p. 180. \square

Remark 8.2. By [6], p. 113, θ_j sends each $R[[q]][q', \dots, q^{(r)}]^\wedge$ into itself hence induces a K -derivation (still denoted by) θ_j on $K[[q, \dots, q^{(r)}]]$ which still satisfies Equations 8.1 (with S_∞^0 replaced by $K[[q]]$).

In the next two Propositions we compute the effect of ∂_j on f^\sharp .

Proposition 8.3. *Assume the hypotheses and notation of Theorem 6.3. Then*

$$(8.2) \quad \partial_2 f^\sharp = p f^{\phi^2}, \quad \partial_1 f^\sharp = -a_p f^\phi, \quad \partial_0 f^\sharp = f.$$

In particular $\partial_{**} f^\sharp = \frac{1}{p}(f^{\phi^2} - a_p f^\phi + p f)$.

Proof. By Lemma 8.1, Theorem 6.3, and Remark 8.2 one has:

$$\begin{aligned} (\partial_2 f^\sharp)_{\infty, \rho} &= \theta_2(f_{\infty, \rho}^\sharp) \\ &= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} \theta_2(\phi^2(q^n)) \\ &= \frac{1}{p} \sum_{n \geq 1} \frac{a_n}{n} p^2 \phi^2(\theta(q^n)) \\ &= p \left(\sum_{n \geq 1} a_n q^n \right)^{\phi^2} \\ &= (p f^{\phi^2})_{\infty, \rho}. \end{aligned}$$

By the δ -expansion principle (Lemma 6.1) we get $\partial_2 f^\sharp = p f^{\phi^2}$. The other equalities are obtained in the same way. \square

In a similar way one proves:

Proposition 8.4. *Assume the hypotheses and notation of Theorem 6.5. Then the following hold.*

- 1) *If p splits in \mathcal{K} then Equations 8.2 hold (with $a_p = 0$).*
- 2) *If p does not split in \mathcal{K} then*

$$(8.3) \quad \partial_1 f^\sharp = f^\phi, \quad \partial_0 f^\sharp = -u f.$$

In particular $\partial_{**} f^\sharp = \frac{1}{p}(f^\phi - p u f)$.

Note that Equations 8.2 and 8.3 together with the condition that f^\sharp is in $M^r(\Gamma_1(N), R_p, 0)$ ($r = 1, 2$) pin down f^\sharp up to an additive constant in R .

9. THE HECKE OPERATOR $T(p)_\infty$

A direct attempt to define the Hecke operator $T(p)$ on δ -modular forms along the lines of Equation 2.1 obviously fails. The “expected” definition of $T(p)$ on arbitrary series in $R[[q, q', \dots, q^{(r)}]]$ is also easily seen to fail. We will define $T(p)_\infty$ on a certain R -submodule of $R[[q, q', \dots, q^{(r)}]]$; then $f_{\infty, \rho}^\sharp$ will be in that submodule and will turn out to be an eigenvector for $T(p)_\infty$ with eigenvalue $a_p(f)$. By considering a slightly different R -module of series we will show that the δ -Fourier expansions $f_{\infty, \rho}^r$ of $f^r = f_{crys}^r$ are also eigenvectors of an appropriate version, $T(p)_{\infty, 2}$, of $T(p)_\infty$ with eigenvalues $p(p+1)$.

Definition 9.1. Let q_1, \dots, q_m be variables and let S_1, \dots, S_m be the fundamental symmetric polynomials in q_1, \dots, q_m ; so

$$S_1 = q_1 + \dots + q_m, \dots, S_m = q_1 \dots q_m.$$

A series

$$G \in R[[q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}]]$$

is δ -symmetric if there exists a series

$$G_{(m)} \in R[[q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}]]$$

such that

$$(9.1) \quad G(q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}) = G_{(m)}(S_1, \dots, S_m, \dots, \delta^r S_1, \dots, \delta^r S_m).$$

(The series $G_{(m)}$ is trivially seen to be unique; cf. [7].)

On the other hand, for any series $F \in R[[q, \dots, q^{(r)}]]$ one can define the series

$$\Sigma_m F := \sum_{j=1}^m F(q_i, \dots, q_i^{(r)}) \in R[[q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}]].$$

The series $\Sigma_m F$ is not δ -symmetric in general. But there are important examples when $\Sigma_m F$ is δ -symmetric; for instance we have:

Lemma 9.2. Let $\mathbf{T} = (T^1, \dots, T^g)$ be a g -tuple of variables, let $\mathcal{F} \in R[[\mathbf{T}_1, \mathbf{T}_2]]^g$ be a formal group law, and let $\psi \in R[[\mathbf{T}, \dots, \mathbf{T}^{(r)}]]$ be such that

$$\psi(\mathcal{F}(\mathbf{T}_1, \mathbf{T}_2), \dots, \delta^r \mathcal{F}(\mathbf{T}_1, \mathbf{T}_2)) = \psi(\mathbf{T}_1, \dots, \mathbf{T}_1^{(r)}) + \psi(\mathbf{T}_2, \dots, \mathbf{T}_2^{(r)})$$

in the ring

$$R[[\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_1^{(r)}, \mathbf{T}_2^{(r)}]].$$

Let $\varphi(q) \in R[[q]]^g$ be a g -tuple of series and let

$$F := \psi(\varphi(q), \dots, \delta^r(\varphi(q))) \in R[[q, \dots, q^{(r)}]].$$

Then $\Sigma_m F$ is δ -symmetric for all $m \geq 2$.

Proof. An easy exercise. Cf. also [7]. □

Corollary 9.3. If $f_{\infty, \rho}^\#$ is as in Theorems 6.3 and 6.5 then $\Sigma_m f_{\infty, \rho}^\#$ is δ -symmetric for all $m \geq 2$.

Proof. By Equations 6.7 and 6.10 $f_{\infty, \rho}^\#$ can be written as

$$\psi(\varphi(q), \dots, \delta^r(\varphi(q)))$$

with ψ and φ as in Lemma 9.2 and we may conclude by Lemma 9.2. □

Definition 9.4. Let $F \in R[[q, \dots, q^{(r)}]]$ be a series such that $G := \Sigma_p F$ is δ -symmetric. Then define the action of the Hecke operator $T(p)_\infty$ on F by

$$(9.2) \quad (T(p)_\infty F) := F(q^p, \dots, \delta^r(q^p)) + G_{(p)}(0, \dots, 0, q, \dots, 0, \dots, 0, q^{(r)}) \in R[[q, \dots, q^{(r)}]].$$

Morally this should correspond to the Hecke action on δ -expansions of weight (of degree) 0.

Remark 9.5. If $G = \Sigma_p F$ is δ -symmetric then so is $G^\phi = \Sigma_p(F^\phi)$ and we have

$$(G^\phi)_{(p)} = (G_{(p)})^\phi.$$

In particular $T(p)_\infty$ commutes with ϕ in the sense that

$$T(p)_\infty(F^\phi) = (T(p)_\infty F)^\phi$$

for any F for which $\Sigma_p F$ is δ -symmetric.

Remark 9.6. If $F = \sum_{n \geq 1} c_n q^n \in R[[q]]$ then $\Sigma_p F$ is δ -symmetric and

$$(9.3) \quad T(p)_\infty F = \sum_{n \geq 1} c_n q^{np} + p \sum_{n \geq 1} c_{np} q^n.$$

Indeed note that if

$$(9.4) \quad q_1^n + \dots + q_p^n = P_n(S_1, \dots, S_p)$$

with P_n a weighted homogeneous polynomial with \mathbf{Z} -coefficients of degree n (with respect to the weights $1, 2, \dots, p$) then $P_n(0, \dots, 0, q)$ is either $mq^{n/p}$ (with $m \in \mathbf{Z}$) or 0 according as p divides n or not. In case $n/p \in \mathbf{Z}$, specializing $q_i \mapsto \zeta_p^i$, $\zeta_p := e^{2\pi i/p}$, we get $S_i \mapsto 0$ for $0 \leq i \leq p-1$ and $S_p \mapsto 1$ so Equation 9.4 yields $p = m$. Equation 9.3 follows. Formula 9.3 is, in some sense, what one would expect the action of $T(p)$ to yield on series of order 0 and weight 0.

Remark 9.7. One can introduce a variant over K of the above definitions. A series

$$G \in K[[q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}]]$$

is $K - \delta$ -symmetric if there exists a series

$$G_{(m)} \in K[[q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}]]$$

such that Equation 9.1 holds. (The series $G_{(m)}$ is, again, trivially seen to be unique; cf. [7].) For any series $F \in K[[q, \dots, q^{(r)}]]$ one can define the series $\Sigma_m F \in K[[q_1, \dots, q_m, \dots, q_1^{(r)}, \dots, q_m^{(r)}]]$ as before. If F is such that $\Sigma_p F$ is $K - \delta$ -symmetric we define $T(p)_\infty F \in K[[q, \dots, q^{(r)}]]$ by the formula 9.2. Then Remarks 9.5 and 9.6 hold verbatim with R replaced by K and the words “ δ -symmetric” replaced by “ $K - \delta$ -symmetric”.

Definition 9.8. A series $F \in R[[q, \dots, q^{(r)}]]$ is an *eigenvector* for $T(p)_\infty$ with *eigenvalue* $\lambda \in R$ if $\Sigma_p F$ is δ -symmetric and

$$(9.5) \quad T(p)_\infty F = \lambda \cdot F.$$

Proposition 9.9. Let $f = \sum a_n q^n$ and f^\sharp be as in Theorems 6.3 and 6.5 respectively. Then $f_{\infty, \rho}^\sharp$ is an eigenvector of $T(p)_\infty$ with eigenvalue $a_p = a_p(f)$.

Proof. By Corollary 9.3 $\Sigma_p f_{\infty, \rho}^\sharp$ is δ -symmetric. Now we think of $f_{\infty, \rho}^\sharp$ as an element of $K[[q, q', q'']]$ or $K[[q, q']]$ respectively and we consider the extension of $T(p)_\infty$ “over K ” discussed in Remark 9.7. Since $T(p)_\infty$ commutes with ϕ it is enough to check that

$$\sum \frac{a_n}{n} q^n$$

is an eigenvector of $T(p)_\infty$ with eigenvalue a_p . This can be checked directly as follows. First note that, by Equations 2.2 we have

$$pa_{n/p} + a_{np} = a_p a_n, \quad n \geq 1.$$

Then, by Remark 9.6, we have

$$\begin{aligned}
T(p)_\infty \left(\sum \frac{a_n}{n} q^n \right) &= \sum \frac{a_n}{n} q^{np} + p \sum \frac{a_{np}}{np} q^n \\
&= \sum \frac{a_{n/p}}{n/p} q^n + \sum \frac{a_{np}}{n} q^n \\
&= \sum \frac{pa_{n/p} + a_{np}}{n} q^n \\
&= a_p \sum \frac{a_n}{n} q^n.
\end{aligned}$$

□

One can develop a variant of $T(p)_\infty$ by allowing it to act on certain series with denominators. We need a preparation. We continue to denote by S_1, \dots, S_p the fundamental symmetric polynomials in q_1, \dots, q_p and we let s_1, \dots, s_p be variables.

Lemma 9.10. *Consider the R -algebras*

$$\begin{aligned}
A &:= R[[s_1, \dots, s_p]][s_p^{-1}]^\wedge [s'_1, \dots, s'_p, \dots, s_1^{(r)}, \dots, s_p^{(r)}]^\wedge, \\
B &:= R[[q_1, \dots, q_p]][q_1^{-1} \dots q_p^{-1}]^\wedge [q'_1, \dots, q'_p, \dots, q_1^{(r)}, \dots, q_p^{(r)}]^\wedge.
\end{aligned}$$

Then the natural algebra map

$$A \rightarrow B, \quad s_j^{(i)} \mapsto \delta^i S_j$$

is injective with torsion free cokernel.

Proof. Let $\sigma_1, \dots, \sigma_{p-1}$ be variables and let $\Sigma_1, \dots, \Sigma_{p-1} \in R[S_1, \dots, S_p]$ be defined by

$$\Sigma_i := q_1^i + \dots + q_p^i.$$

Note that

$$R[\Sigma_1, \dots, \Sigma_{p-1}, S_p] = R[S_1, \dots, S_p].$$

So there is a natural isomorphism $C \simeq A$ where

$$C := R[[\sigma_1, \dots, \sigma_{p-1}, s_p]][s_p^{-1}]^\wedge [\sigma'_1, \dots, \sigma'_{p-1}, s'_p, \dots, \sigma_1^{(r)}, \dots, \sigma_{p-1}^{(r)}, s_p^{(r)}]^\wedge$$

such that the composition $C \rightarrow A \rightarrow B$ is given by

$$\sigma_j^{(i)} \mapsto \delta^i \Sigma_j, \quad s_p^{(i)} \mapsto \delta^i S_p.$$

So it is enough to prove that $C \otimes k \rightarrow B \otimes k$ is injective. We have

$$C \otimes k = k[[\sigma_1, \dots, \sigma_{p-1}, s_p]][s_p^{-1}][\sigma'_1, \dots, \sigma'_{p-1}, s'_p, \dots, \sigma_1^{(r)}, \dots, \sigma_{p-1}^{(r)}, s_p^{(r)}],$$

$$B \otimes k = k[[q_1, \dots, q_p]][q_1^{-1} \dots q_p^{-1}][q'_1, \dots, q'_p, \dots, q_1^{(r)}, \dots, q_p^{(r)}].$$

Now the morphism

$$k[\sigma_1, \dots, \sigma_{p-1}, s_p] = k[s_1, \dots, s_p] \rightarrow k[q_1, \dots, q_p]$$

is finite and flat, and (q_1, \dots, q_p) is the unique maximal ideal lying over

$$(\sigma_1, \dots, \sigma_{p-1}, s_p).$$

Hence the ring homomorphism

$$k[[\sigma_1, \dots, \sigma_{p-1}, s_p]] \rightarrow k[[q_1, \dots, q_p]]$$

is faithfully flat, hence injective, so we have an inclusion $L \subset M$ of their fraction fields. It is enough to show that the map

$$L[\sigma'_1, \dots, \sigma'_{p-1}, s'_p, \dots, \sigma_1^{(r)}, \dots, \sigma_{p-1}^{(r)}, s_p^{(r)}] \rightarrow M[q'_1, \dots, q'_p, \dots, q_1^{(r)}, \dots, q_p^{(r)}]$$

is injective. We will show (and this will end our proof) that for each $i = 0, \dots, r$ the images of

$$(9.6) \quad \delta^i \Sigma_1, \dots, \delta^i \Sigma_{p-1}, \delta^i S_p \in R[q_1, \dots, q_p, \dots, q_1^{(i)}, \dots, q_p^{(i)}]$$

in the ring

$$M[q'_1, \dots, q'_p, \dots, q_1^{(i)}, \dots, q_p^{(i)}]$$

are algebraically independent over

$$M[q'_1, \dots, q'_p, \dots, q_1^{(i-1)}, \dots, q_p^{(i-1)}].$$

Now one checks by induction on i that for all $a = 1, \dots, p-1$,

$$\delta^i \Sigma_a = \sum_{j=1}^p (a q_j^{a(p-1)})^{p^i} q_j^{(i)} + O(i-1) + pO(i),$$

where

$$O(i) \in R[q_1, \dots, q_p, \dots, q_1^{(i)}, \dots, q_p^{(i)}],$$

and $O(i-1)$ has the corresponding meaning. Similarly one has

$$\delta^i S_p = \sum_{j=1}^p q_j^{(i)} (s_p/q_j)^{p^i} + pO(i) + pO(i-1).$$

So the images of the polynomials 9.6 in the ring

$$k[q_1, \dots, q_p, \dots, q_1^{(i)}, \dots, q_p^{(i)}]$$

are (non-homogeneous) linear polynomials in $q_1^{(i)}, \dots, q_p^{(i)}$ with coefficients in

$$k[q_1, \dots, q_p, \dots, q_1^{(i-1)}, \dots, q_p^{(i-1)}].$$

So we need to check that the matrix of the coefficients of $q_1^{(i)}, \dots, q_p^{(i)}$ in the reductions mod p of the polynomials 9.6 is non-singular. But this matrix is the p^i -th power of the matrix

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ 2q_1^p & \cdot & \cdot & \cdot & 2q_p^p \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (p-1)q_1^{(p-2)p} & \cdot & \cdot & \cdot & (p-1)q_p^{(p-2)p} \\ s_p/q_1 & \cdot & \cdot & \cdot & s_p/q_p \end{pmatrix}$$

which is clearly non-singular. \square

Definition 9.11. In the notations of Lemma 9.10, an element $G \in B$ will be called *Laurent δ -symmetric* if it is the image of some element $G_{(p)} \in A$ (which is then unique by Lemma 9.10). For any $F \in R((q))^\wedge[q', \dots, q^{(r)}]^\wedge$ such that

$$\Sigma_p F := \sum_{j=1}^p F(q_j, \dots, q_j^{(r)}) \in B$$

is Laurent δ -symmetric we may define

$$T(p)_{\infty,m}F := F(q^p, \dots, \delta^r(q^p)) + p^m G_{(p)}(0, \dots, 0, q, \dots, 0, \dots, 0, q^{(r)}) \in R[[q, \dots, q^{(r)}]].$$

We write $T(p)_{\infty}F = T(p)_{\infty,0}F$. A series $F \in R((q))^{\wedge}[q', \dots, q^{(r)}]^{\wedge}$ is a *Laurent eigenvector* of $T(p)_{\infty,m}$ with *eigenvalue* $\lambda \in R$ if $\Sigma_p F$ is Laurent δ -symmetric and $T(p)_{\infty,m}F = \lambda \cdot F$. The various values of m morally correspond to the Hecke action on δ -Fourier expansions of δ -modular forms of various weights w (with $\deg(w) = -m$).

Remark 9.12. If $F \in R[[q]][q', \dots, q^{(r)}]^{\wedge}$ then one can consider the following conditions:

- a) $\Sigma_p F$ is δ -symmetric;
- b) $\Sigma_p F$ is Laurent δ -symmetric.

A priori none of these conditions seems to imply the other. On the other hand one can trivially see that f^{\sharp} in Theorems 6.3 and 6.5 is not only δ -symmetric but also Laurent δ -symmetric and a Laurent eigenvector for $T(p)_{\infty}$ with eigenvalue $a_p(f)$.

Proposition 9.13. *For any $r \geq 1$ the δ -Fourier expansion*

$$f_{\infty,\rho}^r \in R((q))^{\wedge}[q', \dots, q^{(r)}]^{\wedge}$$

of the δ -modular form

$$f^r = f_{crys}^r \in M^r(\Gamma_1(N), R, 0)$$

is a Laurent eigenvector for $T(p)_{\infty,2}$ with eigenvalue $p(p+1)$.

Proof. By Lemma 6.2 it is enough to show that Ψ^{ϕ^i} are Laurent eigenvectors for $T(p)_{\infty,2}$ with eigenvalues $p(p+1)$. Now Ψ^{ϕ^i} is Laurent δ -symmetric because

$$\begin{aligned} \sum_{j=1}^p \Psi^{\phi^i}(q_j, \dots, q_j^{(i+1)}) &= \phi^i \left(\sum_{j=1}^p \frac{1}{p} \log \left(1 + p \frac{q_j'}{q_j^p} \right) \right) \\ &= \phi^i \left(\sum_{j=1}^p \frac{1}{p} \log \frac{\phi(q_j)}{q_j^p} \right) \\ &= \phi^i \left(\frac{1}{p} \log \prod_{j=1}^p \frac{\phi(q_j)}{q_j^p} \right) \\ &= \phi^i \left(\frac{1}{p} \log \frac{\phi(s_p)}{s_p^p} \right) \\ &= \Psi^{\phi^i}(s_p). \end{aligned}$$

Moreover, since

$$\Psi^{\phi^i}(q^p, \dots, \delta^{i+1}(q^p)) = \phi^i \left(\frac{1}{p} \log \frac{\phi(q^p)}{q^{p^2}} \right) = p \cdot \Psi^{\phi^i},$$

we have

$$T(p)_{\infty,2}\Psi^{\phi^i} = p \cdot \Psi^{\phi^i} + p^2 \cdot \Psi^{\phi^i} = p(p+1)\Psi^{\phi^i}.$$

□

We end by stating a result (to be proved in a subsequent paper) showing that there is an interesting relationship between (Laurent) δ -symmetry and δ -characters. This result is, in some sense, a “converse” of the existence results for f^{\sharp} in the present paper. We assume, in what follows, that $X_1(N)$ has genus at least 2.

Theorem 9.14. *Fix an embedding $\rho : \mathbf{Z}[1/N, \zeta_N] \rightarrow R_p$ and a δ -modular form $G \in M^r(\Gamma_1(N), R, 0)$. Assume G is δ -holomorphic, G vanishes at ∞ , and $\Sigma_p G_{\infty, \rho}$ is either δ -symmetric or Laurent δ -symmetric. Then $G = \psi \circ \beta$ where $\beta : X_1(N)_R \rightarrow J_1(N)_R$ is the Abel-Jacobi map (corresponding to ∞) and $\psi : J_1(N)_R \rightarrow \mathbf{G}_{a, R}$ is a δ -character. (In particular G is automatically a δ -cusp form.)*

The case when $\Sigma_p G_{\infty, \rho}$ is δ -symmetric follows directly from the main Theorem of [7]. The case when $\Sigma_p G_{\infty, \rho}$ is Laurent δ -symmetric can be proved in an entirely similar way.

Remark 9.15. Given a classical newform $f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{C})$, an embedding $\rho : \mathcal{O}_F[1/MN, \zeta_N] \rightarrow R_p$ (where the cusps are defined over $\mathcal{O}_F[1/M]$), and an embedding $\chi : \mathcal{O}_f^{(Np)} \rightarrow R_p$ one is naturally lead to try to compute the R -module $\mathcal{M} = \mathcal{M}(f, \rho, \chi, r, p)$ of all δ -modular forms $G \in M^r(\Gamma_1(N), R_p, 0)$ satisfying the following properties:

- 1) G is a δ -cusp form (with respect to ρ),
- 2) G belongs to f (outside Np) with character χ and exponent 0,
- 3) $G_{\infty, \rho}$ is an eigenvector (respectively a Laurent eigenvector) of $T(p)_\infty$ with eigenvalue $\chi(a_p)$.

By Proposition 9.14 and Theorem 4.1, $\text{rank } \mathcal{M} \leq rg_1(N)$ where $g_1(N)$ is the genus of $X_1(N)_\mathbf{C}$. One should expect a much better bound for $\text{rank } \mathcal{M}$. Of course, by Theorems 6.3 and 6.5 plus Proposition 9.9, if $g = [K_f : \mathbf{Q}] = 1$ then $f_{\infty, \rho}^\# \in \mathcal{M}$.

REFERENCES

1. Barcau, M: Isogeny covariant differential modular forms and the space of elliptic curves up to isogeny, *Compositio Math.*, **137**, 237-273 (2003)
2. Barcau, M., Buium, A.: Siegel differential modular forms, *International Math Res. Notices* **28**, 1457-1503 (2002).
3. Buium, A.: Differential characters of Abelian varieties over p -adic fields, *Invent. Math.* **122**, 309-340 (1995).
4. Buium, A.: Differential characters and characteristic polynomial of Frobenius, *Crelle J.* **485**, 209-219 (1997).
5. Buium, A.: Differential modular forms, *Crelle J.*, **520**, 95-167 (2000).
6. Buium, A.: Arithmetic Differential Equations. *Math. Surveys and Monographs* **118**, AMS (2005)
7. Buium, A.: Differential characters on curves, preprint.
8. Conrad, B.: The Shimura Construction in weight 2. Appendix to: Ribet, K. A., Stein, W.: Lectures on Serre's conjecture. In: *Arithmetic Algebraic Geometry*, Conrad, B., Rubin K., Eds., IAS/Park City Math Series, Vol. 9, AMS (2001)
9. Darmon, H., Rational points on modular elliptic curves. CBMS No. 101, AMS (2004).
10. Diamond, F., and Im, J.: Modular forms and modular curves. In: *Seminar on Fermat's Last Theorem*, Conference Proceedings, Volume 17, Canadian Mathematical Society, pp. 39-134 (1995).
11. Diamond, F., Shurman, J.: A first course in modular forms. GTM 228, Springer (2005)
12. Goren, E. Z.: Lectures on Hilbert Modular Varieties and Modular Forms. CRM Monograph Series **14**, AMS (2002)
13. Gross, B. H., A tameness criterion for Galois representations associated to modular forms mod p , *Duke Math. J.*, **61**, 2, 445-517 (1990)
14. Manin, Yu. I.: Algebraic curves over fields with differentiation, *Izv. Akad. Nauk SSSR, Ser. Mat.* **22**, 737-756 (1958)
15. Manin, Yu. I.: Parabolic points and zeta functions of modular curves, (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **36**, 19-66 (1972)
16. Katz, N.: p -adic properties of modular schemes and modular forms, LNM 350, Springer, Heidelberg (1973).

17. Katz, N., Mazur, B.: Arithmetic moduli of elliptic curves, Annals of Math. Studies, Princeton Univ. Press (1985)
18. Knapp, A.: Elliptic Curves, Math. Notes, Princeton Univ. Press (1992)
19. Lang, S.: Introduction to Modular forms. Springer, Heidelberg (1976)
20. Ribet, K. A., Stein, W.: Lectures on Serre's conjecture. In: Arithmetic Algebraic Geometry, B. Conrad, K. Rubin Eds., IAS/Park City Math Series, Vol. 9, AMS (2001)
21. Serre, J. P.: Formes modulaires et fonctions zéta p -adiques. In: LNM **350** (1973)
22. Silverman, J. H.: Arithmetic of Elliptic Curves. Springer, Heidelberg, New York (1985)
23. Silverman, J. H.: Advanced Topics in the Arithmetic of Elliptic Curves. Springer, Heidelberg, New York (1994)

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